Gauss Formula for the Julian Date of Passover

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1. Definitions.

Nisan origin is the instant occurring before the beginning of 1 Nisan, in a time difference which is equal to the difference between the new moon, or *molad*, of Tishri and the beginning of 1 Tishri of the following Hebrew year.

Nisan origin of the year 1 AM[1] is 29 Adar, 14 hours, because molad Tishri of the year 2 AM was on Friday, 14 hours, and 1 Tishri was a Saturday.

March origin is the instant occurring one day before the beginning of 1 March. Thus, the March date of a given instant will be in fact the time difference between that instant and March origin.

Note that Nisan origin of the year 1 AM is 33 March, 14 hours, because the Julian date of 1 Nisan 1 AM was 3 April, i.e., 34 March. Here we assume the Julian day begins in the previous evening (at 6 PM), as the Hebrew day does, to simplify the discussion.

Let $H(A)$ denote the Nisan origin and $J(A)$ the March origin of the Hebrew year A. Let T be Nisan origin of the year 1 AM, in March days:

$$T = H(1) - J(1) = 33{:}\,14{:}\,0 = \frac{403}{12}\,. \tag{1}$$

Time intervals are given in the Talmudic units: An hour is divided into 1080 parts, so that d days, h hours and p parts are written as $d{:}\,h{:}\,p = (p/1080 + h)/24 + d$. Let K be 1/19 of the length of the lunar month:

$$K = \frac{29{:}\,12{:}\,793}{19} = \frac{765433}{492480}\,.$$

A cycle of 19 consecutive lunar years contains 235 lunar months, arranged in 12 common years containing 12 months each, and 7 leap years, containing 13 month each. Conventionally, leap years are the 3rd, 6th, 8th, 11th, 14th, 17th, 19th in each cycle. Let L be the average solar excess, i.e., difference in length between a solar year and an average lunar year:

$$L = 365{:}\,6{:}\,0 - 235K = \frac{0{:}\,1{:}\,485}{19} = \frac{313}{98496}\,.$$

Note: Using decimals,

$$T \approx 33.58333333,$$

$$K \approx 1.554241797,$$

$$L \approx 0.003177794022.$$

[1] *anno mundi*.

2. Nisan origin.

The length of a leap lunar year is $13 \cdot 19K = 247K$, while the length of a common lunar year is $12 \cdot 19K = 228K$. Subtracting these quantities from $L + 235K$, the length of the solar year, we get the common solar excess, $L + 7K$, and the (negative) leap solar excess, $L - 12K$.

From these observations we get

$$H(A) - H(1) = 365{:}\,6{:}\,0(A-1) - \Delta(A),$$

where $\Delta(A)$ is the cumulative solar excess. It is given in the following table, with leap years marked by an asterisk:

A	$\Delta(A)$
1	0
2	$L+7K$
*3	$2L-5K$
4	$3L+2K$
5	$4L+9K$
*6	$5L-3K$
7	$6L+4K$
*8	$7L-8K$
9	$8L-K$
10	$9L+6K$
*11	$10L-6K$
12	$11L+K$
13	$12L+8K$
*14	$13L-4K$
15	$14L+3K$
16	$15L+10K$
*17	$16L-2K$
18	$17L+5K$
*19	$18L-7K$
20	$19L$

Each year we add $L + 7K$, unless the year is leap, when we add $L - 12K$ (since we compute in effect the next molad Tishri). In this way, the coefficient of L is incremented continuously, while the coefficient of K is increased by 7 each time, until a moment when it becomes 11 or higher, when it is decreased by 19. Since the lowest possible value of this coefficient is - 8, and this value is obtained at $A = 8 (\mathrm{mod}\ 19)$, we get that the running value is $-8 + (7(A-8))|19$, where $x|k$ is $x - k[x/k]$. Therefore, the coefficient of K is

$$\begin{aligned} -8 + (7(A-8))|19 &= -8 + (7A+1)|19 \\ &= -8 + (18 - (18 - (7A+1))|19) \\ &= 10 - (17 - 7A)|19 \\ &= 10 - (12A+17)|19 \end{aligned}$$

(using the identity $x|k = (k-1) - ((k-1) - x)|k$). Denoting

$$a = (12A+17)|19,$$

we get

$$\Delta(A) = (10-a)K + (A-1)L,$$

or finally

$$H(A) - H(1) = (A-1)365{:}\,6{:}\,0 + (a-10)K - (A-1)L. \tag{2}$$

As an added bonus, we can divide the cycle years into 4 categories, according to the value of the coefficient of K in the cumulative solar excess:

$10-a$	a	$A-1$	A	$A+1$
−8...−2	18...12	common	leap	common
−1...3	11...7	leap	common	common
4...5	6...5	leap	common	leap
6...10	4...0	common	common	leap

3. March origin.

Set

$$J(A) - J(1) = (A-1)365 + \delta(A),$$

where $\delta(A)$ is the number of Julian intercalary days (29 February) between 1 March 1 AM and 1 March of the year A. Since the Hebrew year 1 AM corresponds[2] to the Julian year 3760 BCE, or −3759 CE which gives a remainder of 1 when divided by 4, we obtain that the year A will contain a intercalary day if and only if $A \equiv 0 \pmod 4$. Thus $\delta(A) = [A/4]$, or, denoting

$$b = A|4,$$

we get

$$\delta(A) = A/4 - b/4.$$

Therefore

$$J(A) - J(1) = (A-1)365 + A/4 - b/4,$$

or, finally

$$J(A) - J(1) = (A-1)365{:}\,6{:}\,0 - b/4 + 0{:}\,6{:}\,0. \qquad (3)$$

4. March date of Passover.

Subtracting (3) from (2) and using (1), we get

$$H(A) - J(A) = T + (a-10)K - (A-1)L + b/4 - 0{:}\,6{:}\,0,$$

or

$$H(A) - J(A) = (T - 10K + L) + aK - AL + b/4 - 0{:}\,6{:}\,0.$$

This is the March date of Nisan origin of the Hebrew year A. We add 6 hours to implement the rule that if molad Tishri is at noon or later[3], 1 Tishri is postponed to the following day. Finally we add 14 days to get the March date of 15 Nisan.

Setting

$$m_0 = T - 10K + L + 14 = \frac{3156215}{98496},$$

we get

$$M + m = m_0 + aK - AL + b/4,$$

where M is the integral part and m the fractional part of the right hand side. Unless further exceptions apply (see below), M is the Julian March date of the first day of Passover of the Hebrew year A.

Note: Using decimals,

$$m_0 \approx 32.04409316.$$

[2] At least, its major part containing 1 March.

[3] *Molad Zaken*.

5. Week day of Passover.

Calculating modulo 7, we obtain:

$$J(A) - J(1) \equiv (A-1)365{:}\,6{:}\,0 - b/4 + 0{:}\,6{:}\,0$$

$$\equiv (A-1)1{:}\,6{:}\,0 - b/4 + 0{:}\,6{:}\,0$$

$$\equiv 5A/4 - b/4 - 1$$

$$\equiv A - 1 + (A-b)/4$$

$$\equiv A - 1 + 8(A-b)/4$$

$$\equiv A - 1 + 2(A-b)$$

$$\equiv 3A - 2b - 1$$

$$\equiv 3A + 5b - 1 \pmod 7$$

Since March origin 1 AM was on Friday, we get for M March of the Hebrew year A,

$$c = (M + 3A + 5b + 5)|7.$$

c is the day in the week of M March, with $c = 0$ for Saturday.

6. Exceptions.

In the discussion above, we assumed that 1 Tishri is the day on which molad Tishri has taken place, and established that the Julian date of 15 Nisan is M March. We already mentioned one exception. If molad Tishri is at noon or later, 1 Tishri is postponed to the following day. We implemented this exception by adding 6 hours to Nisan origin. However, there are three more exceptions.

The second exception is the rule that 1 Tishri is excluded from being a Sunday, Wednesday or Friday[4], and is postponed to the following day. To implement this rule, we notice that 15 Nisan and the following 1 Tishri are 152 days apart, i. e., 22 weeks minus 2 days. Thus, 15 Nisan is excluded from being a Friday, Monday of Wednesday, respectively.

The last two exceptions are derived from the previous one, and from a restriction on the length of the Hebrew year. As we have seen, the length of the common lunar year is $12 \cdot 19K = 354{:}\,8{:}\,876$ days, and the length of the leap lunar year is $13 \cdot 19K = 383{:}\,21{:}\,589$ days. Of course, a calendar year must have an integral number of days. Thus, a common Hebrew year has 353, 354 or 355 days[5], while a leap Hebrew year has 383, 384 or 385 days[6].

The third exception follows from restricting the common year to have at most 355 days. Molad Tishri of a common year $A+1$ and its successor are 354:8:876 days apart, i. e., 51 full weeks minus 2:15:204 days. Thus, if molad Tishri of $A+1$, after being moved 6 hours ahead, is on Tuesday, 15 hours and 204 parts or later[7], its successor is on Sunday. Then, 1 Tishri $A+2$ is a Monday, and if 1 Tishri $A+1$ is not postponed from Tuesday (to Thursday, as Wednesday is excluded), the year $A+1$ will have 356 days.

Similarly, the fourth exception follows from restricting the leap year to have at least 383 days. Molad Tishri of a leap year A and its successor are 383:21:589 days apart, i. e., 54 full weeks plus 5:21:589 days. Thus, if molad Tishri of $A+1$, after being moved 6 hours ahead, is on Monday, 21 hours and 589 parts or later[8], its predecessor is on Wednesday. Then, 1 Tishri A is a Thursday, and if 1 Tishri $A+1$ is not

[4] *Adu*.

[5] 12 months, alternating between 30 and 29 days each, give a total of 354 days. This number may increase by adding one to the 29 days of Heshvan, or decrease by subtracting one from the 30 days of Kislev.

[6] The intercalary month, Adar Rishon, has 30 days.

[7] The molad being on Tuesday, 9 hours and 204 parts or later (*Gatrad*).

[8] The molad being on Monday, 15 hours and 589 parts or later (*Betu Takpat*).

postponed from Monday (to Tuesday), the year A will have 382 days.

To implement the last two exceptions, we notice that that 1 Tishri $A+1$ being a Monday or Tuesday implies that 15 Nisan A is a Saturday or Sunday, respectively. Also, if we consider the table in Section 2, we notice that A is leap if $a \geq 12$ and $A+1$ is common if $a \geq 7$.

Thus, setting

$$m_1 = (13 \cdot 19K)|1 = 0{:}\,21{:}\,589 = \frac{23269}{25920},$$

$$m_2 = 1 - (12 \cdot 19K)|1 = 0{:}\,15{:}\,204 = \frac{1367}{2160},$$

we find that the Julian date the first day of Passover is:

- $M+1$ March, if $c = 0$, $a \geq 12$ and $m \geq m_1$,
- $M+2$ March, if $c = 1$, $a \geq 7$ and $m \geq m_2$,
- $M+1$ March, if $c = 2$, 4 or 6,
- M March, otherwise.

Note: Using decimals,

$$m_1 \approx 0.897723765,$$

$$m_2 \approx 0.63287037.$$

7. References.

1. Adler, Cyrus, *Calendar, History of*, in: Singer, Isidore (ed.), The Jewish Encyclopedia, Vol. 3, pp. 498-500. Ktav Publishing House, Inc., New York, 1901.
2. Dershowitz, N. and Reingold, E. M., *Calendrical Calculations*, Software − Practice and Experience, 20 (1990), 899-928.
3. Friedländer, Michael, *Calendar*, in: Singer, Isidore (ed.), The Jewish Encyclopedia, Vol. 3, pp. 501-508. Ktav Publishing House, Inc., New York, 1901.
4. Gauss, Karl Friedrich, *Berechnung Des Jüdishen Osterfestes*, Montaliche Correspondenz zur Beförderung der Erd- und Himmels-Kunde, herausgegeben vom Freiherrn von Zach. Mai 1802. Werke, Vol 6, pp. 80-81.
5. Gauss, Karl Friedrich, *Berechnung Des Neumonds Tisri Für Jedes Jüdische Jahr A*, Handschriftlische Eintragung in Christian Wolf, Elementa matheseos universae, tomus IV. - Von Gauss 1800 erworben. Werke, Vol. 11, pp. 215-218.
6. Resnikoff, Louis A., *Jewish Calendar Calculations*, Scripta Math., 9 (1943), 191-195.

8. Appendix: A Computer Implementation.

```
/*
* Gauss formula for Passover
*
* Arguments:
*    year − Hebrew year (anno mundi)
*    g − boolean flag, 0 for Julian dates, 1 for Gregorian.
*    day − optional pointer to an integer,
*    to return the day−of−the week.
* Return value:
*    March date of the first day of Passover.
*/
```

```
/* Fundamental constants */
#define T  (33. + 14. / 24.)
#define L  ((1.  + 485. / 1080.) / 24. / 19.)
#define K  ((29. + (12. + 793. / 1080.) / 24. )/ 19.)

/* Derived constants */
#define m0 (T − 10. * K + L + 14.)
#define m1 ((21. + 589. / 1080.) / 24.)   /* 13*19*K mod 1 */
#define m2 ((15. + 204. / 1080.) / 24.)   /* 1 − (12*19*K mod 1) */

int
Gauss(int year, int g, int *day)                                        Gauss
{     int a, b, c, M;
      double m;
      a = (12 * year + 17) % 19;
      b = year % 4;
      m = m0 + K * a +  b / 4. − L * year;
      if (m < 0) m−−;
      M = m;
      if (m < 0) m++;
      m −= M;
      switch (c = (M + 3 * year + 5 * b + 5) % 7) {
      case 0:
            if (a >= 12 && m >= m1) {
                  c = 1; M++;
            }
            break;
      case 1:
            if (a >= 7 && m >= m2) {
                  c = 3; M += 2;
            }
            break;
      case 2:
            c = 3; M++;
            break;
      case 4:
            c = 5; M++;
            break;
      case 6:
            c = 0; M++;
            break;
      }
      if (day) *day = c;
      if (g) /* Gregorian Calendar */
            M += (year − 3760) / 100 − (year − 3760) / 400 − 2;
      return M;
}
```