1. **Nonlinear elliptic operators with “discontinuous coefficients”**

Let $\Omega$ be a bounded open set in $\mathbb{R}^N$, $N \geq 2$.

### 1.1. Linear problems.

\[ \begin{cases} -\text{div}(M(x)\nabla u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial \Omega; \end{cases} \tag{1.1} \]

**A:** If $M(x)$ is elliptic and smooth, the Calderon-Zygmund (see also recent results by Haim Brezis) theory states

$$ f \in L^m(\Omega), \quad 1 < m < \infty, \quad \Rightarrow \quad u \in W_0^{1,m^*}(\Omega) $$

**B:** If $M(x)$ is only bounded and elliptic, Guido Stampacchia proved (by duality) the same result for $1 < m < \frac{2N}{N+2}$, that is for infinite energy solutions.

**C:** Moreover, with B-assumptions the above result $f \in L^m(\Omega), \Rightarrow u \in W_0^{1,m^*}(\Omega)$ is false for some values of $m > \frac{2N}{N+2}$ [B-2014].

### 1.2. Nonlinear problems, $p \neq 2$. The simplest example of nonlinear boundary value problem is the Dirichlet problem for the $p$–Laplace operator, with $1 < p < N$, $0 < \alpha \leq a(x) \leq \beta$,

\[ \begin{cases} -\text{div}(a(x)|\nabla u|^{p-2}\nabla u) = f(x), & \text{in } \Omega; \\ u = 0, & \text{on } \partial \Omega; \end{cases} \tag{1.2} \]

The classical theory of nonlinear elliptic equations states that $W_0^{1,p}(\Omega)$ is the natural functional framework to find weak solutions of (1.2), if the function $f$ belongs to the dual space of $W_0^{1,p}(\Omega)$ (see Leray-Lions, Brezis, Browder, Minty).

If the function $f$ does not belong to the dual space of $W_0^{1,p}(\Omega)$, we present our results only on the model problem (1.2), not for general Leray-Lions operators with $p$-coercivity; in particular, we recall the following results (in collaboration with T. Gallouet / past century and new century).

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Theorem 1.1 (Calderon-Zygmund theory for infinite energy solutions).

If \( f \in L^m(\Omega) \), \( \sup \left( 1, \frac{N}{N(p-1)+1} \right) < m < \frac{pN}{pN+p-N} = (p^*)' \),

\( p > 1 + \frac{1}{m} - \frac{1}{N} \), then there exists a distributional solution \( u \in W_0^{1,(p-1)m'}(\Omega) \) of (1.2).

1.3. Existence results in \( W_0^{1,1}(\Omega) \).

Theorem 1.2. Let \( f \in L^m(\Omega) \), \( m = \frac{N}{N(p-1)+1} \), \( 1 < p < 2 - \frac{1}{N} \). Then there exists a distributional solution \( u \in W_0^{1,1}(\Omega) \) of (1.2).

2. The impact of a lower order term of order zero

2.1. Semilinear case. Consider the b.v.p. \((r > 1)\)

\[
\begin{cases}
-\text{div} \left( a(x)|\nabla u|^{p-2} \nabla u \right) + u|u|^{r-2} = f(x) \in L^m(\Omega), \quad \text{in } \Omega; \\
u = 0, \quad \text{on } \partial \Omega.
\end{cases}
\]

The starting point is (at least formally) the “classic” estimate

\[
\|u|u|^{r-2}\|_{L^m(\Omega)} \leq \|f\|_{L^m(\Omega)}
\]

Theorem 2.1. (Cirmi 1995) Let \( r' \leq m < (p^*)' \). Then there exists a weak solution \( u \in W_0^{1,p}(\Omega) \) of (2.1).

Work in progress 2.2. (B-Cirmi) Solutions in \( W_0^{1,1}(\Omega) \).

2.2. Linear case.

\[
\begin{cases}
-\text{div}(M(x)\nabla u) + b(x)u = f(x), \quad \text{in } \Omega; \\
u = 0, \quad \text{on } \partial \Omega.
\end{cases}
\]

Theorem 2.3. (B-Arcoya, 2014) Let \( |f(x)| \leq Qb(x) \in L^1(\Omega) \), \( Q \in \mathbb{R}^+ \). Then there exists a weak, bounded solution \( u \in W_0^{1,2}(\Omega) \) of (2.2).

Work in progress 2.4. (B-Arcoya) \( Q(x) \in L^q(\Omega) \).

\(^1\) (see also the parabolic case)