Proximal point method and projected subgradient method in the presence of computational errors

Alexander J. Zaslavski

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Abstract. We study the convergence of a proximal point method and projected subgradient method in a Hilbert space under the presence of computational errors. Most results known in the literature establish the convergence of optimization methods when computational errors are summable. In the talk we discuss our recent results which establish the convergence of a proximal point method and projected subgradient method for nonsummable computational errors.
Let $X$ be a Hilbert space equipped with an inner product $<\cdot, \cdot>$ which induces the norm $\|\cdot\|$. 

For each function $g : X \to R^1 \cup \{\infty\}$ set 

$$\inf(g) = \inf\{g(y) : y \in X\}.$$ 

We consider the convex optimization problem 

$$(P) \quad \min\{f(x) : x \in X\},$$ 

where $f : X \to R^1 \cup \{\infty\}$ is a convex lower semicontinuous bounded from below function which is not identically $\infty$. One method of solving $(P)$ is to regularize the objective function by using the proximal mapping introduced by Moreau (1965).
Given a real positive number $\lambda$, a proximal approximation of $f$ is defined by

$$f_\lambda(x) = \inf \{ f(u) + (2\lambda)^{-1}||x - u||^2 : u \in X \}.$$  

If the space $X$ is the finite-dimensional Euclidean space $\mathbb{R}^n$ Moreau (1965) proved that the function $f_\lambda$ is convex and differentiable, and when it is minimized it possesses the same set of minimizers and the same optimal value as problem (P). Using these properties, Martinet (1978) introduced the proximal minimization algorithm for solving problem (P) when $X = \mathbb{R}^n$. 
The method is as follows: given an initial point \( x_0 \in X \), a sequence \( \{x_k\}_{k=0}^{\infty} \) is generated by solving

\[
x_k = \arg\min \{ f(x) + (2\lambda_{k-1})^{-1}||x - x_{k-1}||^2 \},
\]

where \( \{\lambda_k\}_{k=0}^{\infty} \) is a sequence of positive numbers.

In our recent work our goal is to establish the convergence of the proximal algorithm in the presence of computational errors for convex minimization problems in infinite-dimensional Hilbert spaces. It should be mentioned that in practice computations introduce numerical errors and if one uses methods in order to solve the auxiliary minimization problems these methods usually provide only approximate solutions of the problems. Clearly, it is very important from the view of practice to study the convergence of iterations of the algorithm in the presence of computational errors.
Most results known in the literature establish the convergence of proximal point methods when computational errors are summable. In the recent paper our goal is to establish the convergence of the proximal algorithm for solving the problem (P) in the presence of computational errors without assuming their summability.
More precisely, it is shown that for a given positive number $\Delta$ there exist a natural number $L$ and $\epsilon > 0$ such that if the computational errors do not exceed $\epsilon$ for any iteration, then the algorithm generates a sequence $\{x_k\}_{k=0}^{\infty}$ such that $f(x_k) \leq \inf(f) + \Delta$ for all integers $k > L$. The constants $L$ and $\epsilon$ depend on $\Delta$. One can see from the statement of this result that $L = c_1 \Delta^{-1}$ and $\epsilon = c_2 \Delta^2$, where $c_1, c_2$ are positive constants which depend on $f$ and do not depend on $\Delta$. 
As we have mentioned before, we assume that $X$ is a Hilbert space equipped with an inner product $\langle \cdot, \cdot \rangle$ which induces the norm $\| \cdot \|$.

Suppose that $f : X \to R^1 \cup \{\infty\}$ is a convex lower semicontinuous function and $a$ is a positive constant such that

$$\text{dom}(f) := \{ x \in X : f(x) < \infty \} \neq \emptyset,$$

(1.1) $f(x) \geq -a$ for all $x \in X$

and that

(1.2) $\lim_{\|x\| \to \infty} f(x) = \infty.$

By (1.1) and (1.2), the set

(1.3) $\text{Argmin}(f) := \{ z \in X : f(z) = \inf(f) \} \neq \emptyset.$

Fix

(1.4) $x^* \in \text{Argmin}(f).$
Let $M$ be any positive number such that

(1.5) $M > \inf(f) + 4$.

By (1.2), there is $M_0 > 1$ such that

(1.6) $f(z) > M + 4$ for all $z \in X$ satisfying $||z|| \geq M_0 - 1$.

Clearly,

(1.7) $||x^*|| < M_0 - 1$.

Assume that

(1.8) $0 < \Lambda_1 < \Lambda_2 \leq M_0^{-2}/2$.

The following result holds.
**Theorem 1.1.** Let

\[(1.9)\quad \lambda_k \in [\Lambda_1, \Lambda_2], \ k = 0, 1, \ldots, \]

\[\Delta \in (0, 1], \text{ a natural number } L \text{ satisfy} \]

\[(1.10)\quad L > 2(4M_0^2 + 1)\Lambda_2 \Delta^{-1} \]

and let a positive number \(\epsilon\) satisfy

\[\epsilon^{1/2}(L + 1)(2\Lambda_1^{-1} + 8M_0\Lambda_1^{-1/2}) \leq 1 \]

\[(1.11)\quad \text{and } \epsilon(L + 1) \leq \Delta/4. \]

Assume that a sequence \(\{x_k\}_{k=0}^{\infty} \subset X\) satisfies

\[(1.12)\quad f(x_0) \leq M \]

and

\[f(x_{k+1}) + 2^{-1}\lambda_k\|x_{k+1} - x_k\|^2 \leq \inf(f + 2^{-1}\lambda_k\| \cdot - x_k\|^2) + \epsilon \]

\[(1.13)\quad \text{for all integers } k \geq 0. \text{ Then for all integers } k > L, \]

\[f(x_k) \leq \inf(f) + \Delta. \]
Theorem 1.1 implies the following result.

**Theorem 1.2** Let

\[ \lambda_k \in [\Lambda_1, \Lambda_2], \; k = 0, 1, \ldots, \]

a natural number \( L \) satisfy

\[ L > (4M_0^2 + 1)2\Lambda_2 \]

and let a positive number \( \bar{\epsilon} \) satisfy

\[ \bar{\epsilon}^{1/2}(L + 1)(2\Lambda_1^{-1} + 8M_0\Lambda_1^{-1/2}) \leq 1 \]

and \( \bar{\epsilon}(L + 1) \leq 1/4 \).

Assume that

\[ \{\epsilon_i\}_{i=0}^{\infty} \subset (0, \bar{\epsilon}), \]

\[ \lim_{i \to \infty} \epsilon_i = 0 \]

and that \( \gamma > 0 \). Then there exists a natural number \( T_0 \) such that for each sequence \( \{x_k\}_{k=0}^{\infty} \subset X \) satisfying

\[ f(x_0) \leq M \]

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and
\[ f(x_{k+1}) + 2^{-1} \lambda_k \| x_{k+1} - x_k \|^2 \leq \inf(f + 2^{-1} \lambda_k \| \cdot - x_k \|^2) + \epsilon_k \]
(1.18)
for all integers \( k \geq 0 \), the inequality
\[ f(x_k) \leq \inf(f) + \gamma \]
holds for all integers \( k > T_0 \).
Since the function \( f \) is convex and lower semi-continuous and satisfies (1.2) Theorem 1.2 easily implies the following result.

**Corollary 1.3** Suppose that all the assumptions of Theorem 1.2 hold and that the sequence \( \{x_k\}_{k=0}^{\infty} \subset X \) satisfies (1.17) and (1.18) for all integers \( k \geq 0 \). Then \( \lim_{k \to \infty} f(x_k) = \inf(f) \) and the sequence \( \{x_k\}_{k=0}^{\infty} \) is bounded. Moreover, it possesses a weakly convergent subsequence and the limit of any weakly convergent subsequence of \( \{x_k\}_{k=0}^{\infty} \) is a minimizer of \( f \).

Problem (P) is called well-posed if the function \( f \) possesses a unique minimizer which is a limit in the norm topology of any minimizing sequence of \( f \).

Corollary 1.3 easily implies the following result.
Corollary 1.4 Suppose that problem (P) is well-posed, all the assumptions of Theorem 1.2 hold and that the sequence \( \{x_k\}_{k=0}^{\infty} \subset X \) satisfies (1.17) and (1.18) for all integers \( k \geq 0 \). Then \( \{x_k\}_{k=0}^{\infty} \) converges in the norm topology to a unique minimizer of \( f \).

Note that that most problems of type (P) (in the sense of Baire category) are well-posed.
Auxiliary results

**Lemma 2.1** Assume that

\begin{equation}
\lambda_k \in [\Lambda_1, \Lambda_2], \ k = 0, 1, \ldots
\end{equation}

and that a sequence \(\{x_k\}_{k=0}^\infty\) satisfies

\begin{equation}
f(x_0) \leq M,
\end{equation}

\[
f(x_{k+1}) + 2^{-1} \lambda_k \|x_{k+1} - x_k\|^2
\]

\[
\leq \inf(f + 2^{-1} \lambda_k \|-x_k\|^2) + 1
\]

for all integers \(k \geq 0\). Then \(\|x_k\| \leq M_0\) for all integers \(k \geq 0\).
Proof. By (2.2) and (1.6), $\|x_0\| \leq M_0$. Assume that an integer $k \geq 0$ and that

(2.4) $\|x_k\| \leq M_0$.

By (2.3), (2.1), (1.7), (1.8), (1.3), (1.4), (1.5) and (2.4),

$$f(x_{k+1}) \leq f(x^*) + 2^{-1}\lambda_k \|x^* - x_k\|^2 + 1$$

$$\leq f(x^*) + 2^{-1} \Lambda 2(2M_0)^2 + 1$$

$$\leq f(x^*) + 2 = \inf(f) + 2 < M.$$  

Combined with (1.6) this implies that $\|x_{k+1}\| \leq M_0$. Thus we showed by induction that (2.4) holds for all integers $k \geq 0$. Lemma 2.1 is proved.
Lemma 2.2 Assume that

\( \lambda_k \in [\Lambda_1, \Lambda_2], \ k = 0, 1, \ldots, \)
\( \epsilon_k \in (0, 1], \ k = 0, 1, \ldots, \) a sequence \( \{x_k\}_{k=0}^{\infty} \subset X \) satisfies

\[
(2.5) \quad f(x_0) \leq M
\]

and that for all integers \( k \geq 0, \)

\[
(2.6) \quad f(x_{k+1}) + 2^{-1}\lambda_k \|x_{k+1} - x_k\|^2
\]

\[
(2.7) \quad \leq \inf(f + 2^{-1}\lambda_k \| \cdot - x_k \|^2) + \epsilon_k.
\]

Then the following assertions hold.

1. For all integers \( k \geq 0, \)

\[
(2/\lambda_k)(f(x_{k+1}) - f(x^*)) + \|x_{k+1} - x_k\|^2
\]

\[
\leq 2\epsilon_k\Lambda_1^{-1} + \|x_k - x^*\|^2
\]

\[
-\|x_{k+1} - x^*\|^2 + 8M_0(\epsilon_k\Lambda_1^{-1})^{1/2}.
\]
2. For all pairs of natural numbers \( m > n \),
\[
\sum_{i=n}^{m} 2\Lambda^{-1}_2(f(x_i) - f(x^*)) + \sum_{i=n}^{m} ||x_{i-1} - x_i||^2
\leq 4M_0^2 + \sum_{i=n-1}^{m-1} [2\Lambda^{-1}_1\epsilon_i + 8M_0(\epsilon_i\Lambda^{-1}_1)^{1/2}].
\]

**Proof** By (2.6), (2.7) and Lemma 2.1,

(2.8) \( ||x_k|| \leq M_0 \) for all integers \( k \geq 0 \).

By (2.7), for all integers \( k \geq 0 \),

(2.9) \( f(x_{k+1}) \leq f(x_k) + \epsilon_k \).
We will prove Assertion 1. Let an integer $k \geq 0$. There exists $y_{k+1} \in X$ such that

$$f(y_{k+1}) + 2^{-1}\lambda_k||y_{k+1} - x_k||^2$$

(2.10) $\leq f(x) + 2^{-1}\lambda_k||x - x_k||^2$ for all $x \in X$.

We estimate $||x_{k+1} - y_{k+1}||$. Put

(2.11) $z = 2^{-1}(x_{k+1} + y_{k+1})$.

It is clear that

$$2^{-1}||y_{k+1} - x_k||^2 + 2^{-1}||x_{k+1} - x_k||^2 - ||2^{-1}(x_{k+1} + y_{k+1}) - x_k||^2 = 2^{-1}||y_{k+1}||^2 + 2^{-1}||x_k||^2 - < y_{k+1}, x_k > + 2^{-1}||x_{k+1}||^2 + ||x_k||^2/2$$

$$- < x_{k+1}, x_k > - ||x_k||^2 + < x_k, x_{k+1} > + y_{k+1} > - ||2^{-1}(y_{k+1} + x_{k+1})||^2 = 2^{-1}||y_{k+1}||^2 + 2^{-1}||x_{k+1}||^2$$

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\[-\|2^{-1}(y_{k+1} + x_{k+1})\|^2\]

\[(2.12) \quad = \|2^{-1}(y_{k+1} - x_{k+1})\|^2.
\]

By (2.11), convexity of \(f\), (2.12), (2.10) and (2.7),

\[
f(z) + 2^{-1}\lambda_k\|z - x_k\|^2
\]

\[
\leq 2^{-1}f(x_{k+1}) + 2^{-1}f(y_{k+1})
\]

\[
+ 2^{-1}\lambda_k(2^{-1}\|y_{k+1} - x_k\|^2
\]

\[
+ 2^{-1}\|x_{k+1} - x_k\|^2
\]

\[
- \|2^{-1}(y_{k+1} - x_{k+1})\|^2
\]

\[
\leq \inf\{f(x) + 2^{-1}\lambda_k\|x - x_k\| \colon x \in X\}
\]

\[
+ 2^{-1}\epsilon_k - 2^{-1}\lambda_k\|2^{-1}(y_{k+1} - x_{k+1})\|^2.
\]

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Together with (2.1) this inequality implies that
\[ \|2^{-1}(y_{k+1} - x_{k+1})\|^2 \leq \epsilon_k \Lambda_1^{-1} \]
and that
\[ (2.13) \quad \|y_{k+1} - x_{k+1}\| \leq 2(\epsilon_k \Lambda_1^{-1})^{1/2}. \]
Now we estimate \( f(x^*) - f(x_{k+1}) \). By (2.10),
\[ 0 \in \partial f(y_{k+1}) + \lambda_k (y_{k+1} - x_k) \]
and for all \( u \in X \),
\[ f(u) - f(y_{k+1}) \]
\[ (2.14) \quad \geq \lambda_k < x_k - y_{k+1}, u - y_{k+1} >. \]
In view of (2.14),
\[ f(x^*) - f(y_{k+1}) \]
\[ (2.15) \quad \geq \lambda_k < x_k - y_{k+1}, x^* - y_{k+1} >. \]
By (2.7),
\[ f(x_{k+1}) + 2^{-1} \lambda_k \|x_{k+1} - x_k\|^2 \]
\[ \leq f(y_{k+1}) + 2^{-1} \lambda_k \|y_{k+1} - x_k\|^2 + \epsilon_k. \]
and

\[
f(y_{k+1}) - f(x_{k+1}) \\
\geq 2^{-1} \lambda_k (||x_{k+1} - x_k||^2 - ||y_{k+1} - x_k||^2) - \epsilon_k.
\]

Combined with (2.15) this implies that

\[
f(x^*) - f(x_{k+1}) \\
= f(x^*) - f(y_{k+1}) + f(y_{k+1}) - f(x_{k+1}) \\
\geq \lambda_k < x_k - y_{k+1}, x^* - y_{k+1} > \\
+f(y_{k+1}) - f(x_{k+1})
\]

\[
= 2^{-1} \lambda_k [||y_{k+1} - x^*||^2 \\
-||x_k - x^*||^2 + ||x_k - y_{k+1}||^2] + f(y_{k+1}) - f(x_{k+1}) \\
\geq 2^{-1} \lambda_k [||y_{k+1} - x^*||^2 \\
-||x_k - x^*||^2 + ||x_k - x_{k+1}||^2] - \epsilon_k.
\]
By (2.10), (2.5), (2.8), (1.7), (1.8) and (1.5), for all integers \(q \geq 1\),

\[
f(y_q) \leq f(x^*) + 2^{-1}\Lambda_2\|x^* - x_q\|^2 \\
\leq f(x^*) + 1 < M.
\]

Together with (1.6) this inequality implies that

(2.17) \[\|y_q\| \leq M_0, \quad q = 1, 2, \ldots\]

Now we use (2.16) and (2.17) and obtain an estimation of \(f(x^*) - f(x_{k+1})\) without terms which contain \(y_{k+1}\).
By (2.8) and (2.13),

\[ ||x_k - y_{k+1}||^2 \]
\[ = ||(x_k - x_{k+1}) - (y_{k+1} - x_{k+1})||^2 \]
\[ = ||x_k - x_{k+1}||^2 + ||y_{k+1} - x_{k+1}||^2 \]
\[ - 2 < x_k - x_{k+1}, y_{k+1} - x_{k+1} > \]

\[ \geq ||x_k - x_{k+1}||^2 \]
\[ - 2||x_k - x_{k+1}|| ||y_{k+1} - x_{k+1}|| \]
\[ \geq ||x_k - x_{k+1}||^2 \]

(2.18) \[ -8M_0(\epsilon_k \Lambda_1^{-1})^{1/2}. \]
It follows from (2.8), (1.7) and (2.13) that

\[ \| y_{k+1} - x^* \|^2 = \| (x_{k+1} - x^*) + (y_{k+1} - x_{k+1}) \|^2 = \| x_{k+1} - x^* \|^2 \]

\[ + \| y_{k+1} - x_{k+1} \|^2 + 2 < x_{k+1} - x^*, y_{k+1} - x_{k+1} > \geq \| x_{k+1} - x^* \|^2 \]

(2.19) \quad -8M_0(\epsilon_k\Lambda_1^{-1})^{1/2}.

By (2.16) and (2.19),

\[ f(x^*) - f(x_{k+1}) \geq -\epsilon_k + 2^{-1}\lambda_k [\| x_{k+1} - x^* \|^2 \]

\[ -\| x^* - x_k \|^2 + \| x_k - x_{k+1} \|^2 - 8M_0(\epsilon_k\Lambda_1^{-1})^{1/2}], \]
\[ f(x_{k+1}) - f(x^*) \]
\[ + 2^{-1} \lambda_k \|x_k - x_{k+1}\|^2 \]
\[ \leq \epsilon_k + 2^{-1} \lambda_k \left[ \|x_k - x^*\|^2 - \|x_{k+1} - x^*\|^2 \right] \]
\[ + 2^{-1} \lambda_k 8M_0(\epsilon_k \Lambda_1^{-1})^{1/2} \]

and in view of (2.5),
\[ (2/\lambda_k)(f(x_{k+1}) - f(x^*)) \]
\[ + \|x_k - x_{k+1}\|^2 \]
\[ \leq 2\epsilon_k \Lambda_1^{-1} + \|x_k - x^*\|^2 \]
\[ - \|x_{k+1} - x^*\|^2 + 8M_0(\epsilon_k \Lambda_1^{-1})^{1/2}. \]

Thus Assertion 1 is proved.
Let us prove Assertion 2. By Assertion 1, (2.5), (1.7) and (2.8), for all pairs of natural numbers \( m > n \),

\[
\sum_{i=n}^{m} \left( \frac{2}{\Lambda_2} (f(x_i) - f(x^*)) \right) + \sum_{i=n}^{m} ||x_{i-1} - x_i||^2 \\
\leq ||x_{n-1} - x^*||^2 \\
+ \sum_{i=n-1}^{m-1} \left[ 2\epsilon_i \Lambda_1^{-1} + 8M_0(\epsilon_i \Lambda_1^{-1})^{1/2} \right] \\
\leq 4M_0^2 + \sum_{i=n-1}^{m-1} \left[ 2\Lambda_1^{-1} \epsilon_i + 8M_0(\epsilon_i \Lambda_1^{-1})^{1/2} \right].
\]

Assertion 2 is proved. This completes the proof of Lemma 2.2.
Proof of Theorem 1.1

By (1.9), (1.10), (1.11), (1.12), (1.13) and Lemma 2.2 applied for a natural number \( n \) and \( m = n + L \),

\[
\sum_{i=n}^{n+L} (2/\Lambda_2)(f(x_i) - f(x^*)) \leq 4M_0^2 + \sum_{i=n-1}^{n-1+L} [2\Lambda_1^{-1}\epsilon + 8M_0(\epsilon\Lambda_1^{-1})^{1/2}]
\]

\[
\leq 4M_0^2 + (L + 1)\epsilon^{1/2}[2\Lambda_1^{-1} + 8M_0\Lambda_1^{-1/2}]
\]

(3.1) \( \leq 4M_0^2 + 1 \).

Let \( n \) be a natural number. By (3.1),

\[
(L+1)2\Lambda_2^{-1} \min\{f(x_i) - f(x^*) : i = n, \ldots, n+L\} \leq 4M_0^2 + 1
\]
and in view of (1.10),

\[
\min \{ f(x_i) - f(x^*) : i = n, \ldots, n + L \}
\]

(3.2) \( \leq (4M_0^2 + 1)(L + 1)^{-1}2^{-1}1 < \Delta / 4. \)

Since (3.2) holds for any natural number \( n \) there exists a strictly increasing sequence of natural numbers \( \{ S_i \}_{i=1}^{\infty} \) such that

\[ S_1 \in \{ 1, \ldots, 1 + L \}, \]

(3.3) \( S_{i+1} - S_i \in [1, 1 + L], \; i = 1, 2, \ldots, \)

(3.4) \( f(x_{S_i}) - f(x^*) \leq \Delta / 4, \; i = 1, 2, \ldots. \)

Let an integer \( j \geq L + 1. \) By (3.3) there exists a natural number \( i \) such that

\[ S_i \leq j < S_{i+1} \]

and

(3.5) \( j - S_i \leq L + 1. \)
In view of (1.13) for all integers $k \geq 0$,

$$f(x_{k+1}) \leq f(x_k) + \epsilon.$$  

Together with (3.5), (3.4) and (1.11) this implies that

$$f(x_j) \leq f(x_{S_i}) + \epsilon(L+1) \leq f(x^*) + \Delta/4 + \Delta/4.$$  

Theorem 1.1 is proved.

Proof of Theorem 1.2

By Theorem 1.1 the following property holds:

(P1) Let a sequence $\{x_k\}_{k=0}^{\infty} \subset X$ satisfy

$$f(x_0) \leq M,$$

$$f(x_{k+1}) + 2^{-1}\lambda_k \|x_{k+1} - x_k\|^2$$

$$\leq \inf(f + 2^{-1}\lambda_k \| \cdot - x_k\|^2) + \bar{\epsilon}, \; k = 0, 1, \ldots.$$  

Then

$$f(x_k) \leq \inf(f) + 1$$

for all integers $k > L$.  

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By Theorem 1.1 there exist $\delta \in (0, \bar{\epsilon})$ and a natural number $L_0$ such that the following property holds:

(P2) For each sequence $\{y_i\}_{i=0}^{\infty} \subset X$ which satisfies

$$f(y_0) \leq \inf(f) + 1,$$

$$f(y_{k+1}) + 2^{-1}\lambda_k\|y_{k+1} - y_k\|^2 \leq \inf(f + 2^{-1}\lambda_k\cdot -y_k\|^2) + \delta$$

for all integers $k \geq 0$ we have

$$f(y_k) \leq \inf(f) + \gamma$$

(4.1) for all integers $k \geq L_0$.

(Here $\gamma$ is as in the statement of the theorem.)

By (1.16), there is a natural number $L_1$ such that

(4.2) $\epsilon_k < \delta$ for all natural numbers $k \geq L_1$.  

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Choose a natural number

\begin{equation}
T_0 > L_0 + L_1 + L.
\end{equation}

Assume that a sequence \( \{x_i\}_{i=0}^\infty \subset X \) satisfies (1.17) and (1.18). By (P1), (1.17), (1.18) and (1.16),

\[ f(x_k) \leq \inf(f) + 1 \]

\begin{equation}
\text{for all integers } k > L.
\end{equation}

For each integer \( k \geq 0 \) set

\begin{equation}
y_k = x_k + L + L_1.
\end{equation}

By (4.5) and (4.4),

\begin{equation}
f(y_0) \leq \inf(f) + 1.
\end{equation}

It follows from (1.18), (4.5) and (4.2) that for all integers \( k \geq 0 \),

\[ f(y_{k+1}) + 2^{-1}\lambda_k \|y_{k+1} - y_k\|^2 \]

\begin{equation}
\leq \inf(f + 2^{-1}\lambda_k \| \cdot - y_k \|^2) + \delta.
\end{equation}
By (4.6), (4.7) and (P2),

(4.8) \( f(y_k) \leq \inf(f) + \gamma \) for all integers \( k \geq L_0 \).

Together with (4.5) and (4.3) this inequality implies that

\[ f(x_k) \leq \inf(f) + \gamma \] for all integers \( k > T_0 \).

Theorem 1.2 is proved.
The projected subgradient method for nonsmooth convex optimization in the presence of computational errors

We study convergence of the projected subgradient method for constrained convex optimization in a Hilbert space. Our goal is to obtain an $\epsilon$-approximate solution of the problem in the presence of computational errors, where $\epsilon$ is a given positive number. The results which we obtain are important from the point of view of practice because computations always introduce numerical errors.
The study of convergence of projected subgradient methods for convex optimization in infinite-dimensional spaces and in the presence of computational errors has recently been a rapidly growing area of research. In particular in Ya. I. Alber, A. N. Iusem and M. V. Solodov, Math. Programming, 81, 1998. it is considered an extension of the projected subgradient method for constrained convex optimization in a Hilbert space which consists of a step in the direction opposite to an $\epsilon_k$-subgradient of the objective at a current iterate, followed by an orthogonal projection onto the feasible set. The normalized stepsizes $\alpha_k$ are required to satisfy $\sum_{k=0}^{\infty} \alpha_k = \infty$ and $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$, and $\epsilon_k$ is taken so that $\epsilon_k \leq \mu \alpha_k$, where $\mu$ is a positive constant. Under these assumptions it is shown that the sequence generated by the method weakly converges to a minimizer. It should be mentioned that work is a very important step in the study of projected subgradient methods because it significantly improves all convergence results previously known for such methods.
In our recent work we also study convergence of the projected subgradient method for constrained convex optimization in a Hilbert space but our goal is to obtain an $\epsilon$-approximate solution of the problem in the presence of computational errors, where $\epsilon$ is a given positive number. Clearly, in practice it is sufficient to find an $\epsilon$-approximate solution instead of constructing a minimizing sequence. On the other hand in practice computations introduce numerical errors and if one uses methods in order to solve minimization problems these methods usually provide only approximate solutions of the problems. Clearly, it is very important from the point of view of practice to study the convergence of iterations of algorithms in the presence of computational errors. It turns out that in order to meet our goal we need less restrictive assumptions than those of Ya. I. Alber, A. N. Iusem and M. V. Solodov, Math. Programming, 81, 1998.
For example, we do not need to assume that the stepsizes $\alpha_k$ to satisfy $\sum_{k=0}^{\infty} \alpha_k^2 < \infty$ and that $\epsilon_k$ is taken so that $\epsilon_k \leq \mu \alpha_k$. Instead of it we assume that $\epsilon_k \leq \mu$, where $\mu$ is a positive constant which depends on $\epsilon$.

Let $(X, < \cdot, \cdot >)$ be a Hilbert space with a inner product $< \cdot, \cdot >$ which induces a complete norm $\| \cdot \|$. 

For each $x \in X$ and each nonempty set $A \subset X$ put

$$\rho(x, A) = \inf\{\|x - y\| : y \in A\}.$$ 

For each $x \in X$ and each $r > 0$ set

$$B(x, r) = \{y \in X : \|x - y\| \leq r\}.$$ 

Assume that $f : X \rightarrow R^1$ is a convex continuous function which is Lipschitz on all bounded subsets of $X$. 

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For each $x \in X$ and each $\epsilon > 0$ let 

$$\partial f(x) = \{l \in X : f(y) - f(x) \geq \langle l, y - x \rangle \text{ for all } y \in X\}$$

be the subdifferential of $f$ at $x$ and let 

$$\partial_{\epsilon} f(x) = \{l \in X : f(y) - f(x) \geq \langle l, y - x \rangle - \epsilon \text{ for all } y \in X\}$$

be the $\epsilon$-subdifferential of $f$ at $x$.

Let $C$ be a closed (in the normed topology) nonempty subset of $X$.

Assume that 

$$\lim_{\|x\| \to \infty} f(x) = \infty.$$ 

It means that for each $M_0 > 0$ there is $M_1 > 0$ such that if $x \in X$ satisfies $\|x\| \geq M_1$, then $f(x) > M_0$.

Set 

$$\inf(f; C) = \inf\{f(z) : z \in C\}.$$
Since $f$ is Lipschitz on bounded subsets of $X$, $\inf(f; C')$ is finite.

Put

$$C_{\min} = \{x \in C : f(x) = \inf(f; C')\}.$$  

It is well-known that if $C$ is convex, then $C_{\min}$ is nonempty. It is clear that $C_{\min} \neq \emptyset$ if $X$ is finite-dimensional.

We assume that

$$C_{\min} \neq \emptyset.$$  

It is clear that $C_{\min}$ is a closed subset of $X$.  

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It is not difficult to see that the following proposition holds.

**Proposition 1.1** If $X$ is finite-dimensional, then for each $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in C$ satisfies $f(x) \leq \inf(f; C) + \epsilon$, then $\rho(x, C_{\min}) \leq \epsilon$.

We suppose that the following assumption holds.

(A1) For each $\epsilon > 0$ there exists $\delta > 0$ such that if $x \in C$ satisfies $f(x) \leq \inf(f; C) + \delta$, then $\rho(x, C_{\min}) \leq \epsilon$.

It is well-known that the following proposition holds.

**Proposition 1.2** Assume that $C$ is convex. Then for each $x \in X$ there is a unique point $P_C(x) \in C$ satisfying

$$||x - P_Cx|| = \inf\{||y - x|| : y \in C\}.$$ 

Moreover,

$$||P_Cy - P_Cz|| \leq ||y - z||$$

for all $y, z \in X$. 

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We suppose that the following assumption holds.

(A2) There exists a continuous mapping $P_C : X \to X$ such that $P_C(X) = C$, $P_Cx = x$ for all $x \in C$ and

$$||x - P_Cy|| \leq ||x - y||$$

for all $x \in C$ and all $y \in X$.

For each $\epsilon \in (0, \infty)$ put

$$\phi(\epsilon) = \sup\{\delta \in (0, 1] :$$

if $x \in C$ satisfies $f(x) \leq \inf(f; C) + \delta$,

then $\rho(x, C_{min}) \leq \min\{1, \epsilon\}$.}

By (A1), $\phi(\epsilon)$ is well-defined for all $\epsilon > 0$. 
In our recent work we establish the following two results.

**Theorem 1.1** Let \( \{\alpha_i\}_{i=0}^{\infty} \subset (0, 1] \) satisfy
\[
\lim_{i \to \infty} \alpha_i = 0,
\]
\[
\sum_{i=1}^{\infty} \alpha_i = \infty
\]
and let \( M, \epsilon > 0 \). Then there exist a natural number \( n_0 \) and \( \delta > 0 \) such that the following assertion holds.

Assume that an integer \( n \geq n_0 \),
\[
\{x_k\}_{k=0}^{n} \subset X, \quad \|x_0\| \leq M,
\]
\[
v_k \in \partial_{\delta} f(x_k) \setminus \{0\},
\]
\[
k = 0, 1, \ldots, n - 1,
\]
\[
\{\eta_k\}_{k=0}^{n-1}, \{\xi_k\}_{k=0}^{n-1} \subset B(0, \delta),
\]
and that for $k = 0, \ldots, n - 1$, 

$$x_{k+1} = P_C(x_k - \alpha_k \|v_k\|^{-1}v_k - \alpha_k \xi_k) - \alpha_k \eta_k.$$  

Then the inequality $\rho(x_k, C_{\min}) \leq \epsilon$ holds for all integers $k$ satisfying $n_0 \leq k \leq n$. 
**Theorem 1.2** Let $M, \epsilon > 0$. Then there exists $\beta_0 \in (0, 1)$ such that for each $\beta_1 \in (0, \beta_0)$ there exist a natural number $n_0$ and $\delta > 0$ such that the following assertion holds.

Assume that an integer $n \geq n_0$,

$$\{x_k\}_{k=0}^{n} \subset X, \, \|x_0\| \leq M,$$

$$v_k \in \partial_{\delta} f(x_k) \setminus \{0\},$$

$$k = 0, 1, \ldots, n - 1,$$

$$\{\alpha_k\}_{k=0}^{n-1} \subset [\beta_1, \beta_0],$$

$$\{\eta_k\}_{k=0}^{n-1}, \, \{\xi_k\}_{k=0}^{n-1} \subset B(0, \delta)$$

and that for $k = 0, \ldots, n - 1$,

$$x_{k+1} = P_C(x_k - \alpha_k \|v_k\|^{-1}v_k - \alpha_k \xi_k) - \eta_k.$$

Then the inequality $\rho(x_k, C_{min}) \leq \epsilon$ holds for all integers $k$ satisfying $n_0 \leq k \leq n$. 

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