THE COMBINATORIAL INVERSE EIGENVALUE PROBLEM: COMPLETE GRAPHS AND SMALL GRAPHS WITH STRICT INEQUALITY∗

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Abstract. Let $G$ be a simple undirected graph on $n$ vertices and let $\mathcal{S}(G)$ be the class of real symmetric $n \times n$ matrices whose nonzero off-diagonal entries correspond exactly to the edges of $G$. Given $2n - 1$ real numbers $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \cdots \geq \lambda_{n-1} \geq \mu_{n-1} \geq \lambda_n$, and a vertex $v$ of $G$, the question is addressed of whether or not there exists $A \in \mathcal{S}(G)$ with eigenvalues $\lambda_1, \ldots, \lambda_n$ such that $A(v)$ has eigenvalues $\mu_1, \ldots, \mu_{n-1}$, where $A(v)$ denotes the matrix with the $v$th row and column deleted. General results that apply to all connected graphs $G$ are given first, followed by a complete answer to the question for $K_n$. Since the answer is constructive it can be implemented as an algorithm; a Mathematica code is provided to do so. Finally, for all connected graphs on 4 vertices it is shown that the answer is affirmative if all six inequalities are strict.

Key words. Graph, Interlacing inequalities, Inverse eigenvalue problem, Symmetric matrix.

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