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GRAPHS WHOSE MINIMAL RANK IS TWO*

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Abstract. Let F be a field, $G = (V, E)$ be an undirected graph on n vertices, and let $S(F, G)$ be the set of all symmetric $n \times n$ matrices whose nonzero off-diagonal entries occur in exactly the positions corresponding to the edges of G. For example, if G is a path, $S(F, G)$ consists of the symmetric irreducible tridiagonal matrices. Let $\mathrm{mr}(F, G)$ be the minimum rank over all matrices in $S(F, G)$. Then $\mathrm{mr}(F, G) = 1$ if and only if G is the union of a clique with at least 2 vertices and an independent set. If F is an infinite field such that $\mathrm{char}\, F \neq 2$, then $\mathrm{mr}(F, G) \leq 2$ if and only if the complement of G is the join of a clique and a graph that is the union of at most two cliques and any number of complete bipartite graphs. A similar result is obtained in the case that F is an infinite field with $\mathrm{char}\, F = 2$. Furthermore, in each case, such graphs are characterized as those for which 6 specific graphs do not occur as induced subgraphs. The number of forbidden subgraphs is reduced to 4 if the graph is connected. Finally, similar criteria is obtained for the minimum rank of a Hermitian matrix to be less than or equal to two. The complement is the join of a clique and a graph that is the union of any number of cliques and any number of complete bipartite graphs. The number of forbidden subgraphs is now 5, or in the connected case, 3.

Key words. Rank 2, Minimum rank, Symmetric matrix, Forbidden subgraph, Bilinear symmetric form.

AMS subject classifications. 05C50, 05C75, 15A03, 15A57.

1. Introduction.

Given any field F, let $\mathrm{char}\, F$ be the characteristic of F, let F^* be the nonzero elements of F, and let $F^n = \{[x_1, \ldots, x_n]^t | x_1, \ldots, x_n \in F\}$. Given a (simple, undirected) graph $G = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$, let $S(F, G)$ be the set of all symmetric $n \times n$ matrices $A = [a_{ij}]$ with entries in F such that $a_{ij} \neq 0$, $i \neq j$, if and only if $ij \in E$. There is no restriction on the diagonal entries of A. We study the problem of minimizing the rank for all $A \in S(F, G)$. Let

$$\mathrm{mr}(F, G) = \min\{\mathrm{rank}\, A | A \in S(F, G)\}.$$

If $F = \mathbb{R}$, we abbreviate $\mathrm{mr}(F, G)$ and $S(F, G)$ to $\mathrm{mr}(G)$ and $S(G)$, respectively. Then minimizing the rank is equivalent to maximizing the multiplicity of an eigenvalue of $A \in S(G)$. It is easy to see that the maximum multiplicity of an eigenvalue of $A \in S(G)$ is $n - \mathrm{mr}(G)$. This problem is completely solved for trees by Duarte and Johnson [8]. Several additional results have been obtained for the multiplicities of eigenvalues of a matrix in $S(G)$ in the case G is a tree ([10], [9], [11]). Results in a different direction were obtained by Colin de Verdière, Lovász, Schrijver, and van der

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Holst ([2, 12, 13, 5]). They are associated with a graph parameter $\mu(G)$ introduced in [2] and defined in terms of the multiplicity of the second smallest eigenvalue of matrices in $S(G)$ with a particular sign pattern and satisfying a certain transversality condition. Many of these results are gathered together in the survey paper [7]. Colin de Verdière introduced another graph parameter $\nu(G)$ in [3] more in line with this paper. It is the largest nullity attained by a positive semidefinite matrix in $S(G)$ that also satisfies a transversality condition. Van der Holst considered the parameter $\tau(G)$ in [6], the largest nullity of an n-by-n Hermitian positive semidefinite matrix with graph G (without the transversality condition), and building upon the results in [3] was able to characterize those graphs G with $\tau(G) \leq 1$ and $\tau(G) \leq 2$. Later, we will also consider special cases of our results when $F = \mathbb{R}$. We let $S_+(G)$ be the set of positive semidefinite matrices in $S(G)$ and let

$$\mathrm{mr}_+(G) = \min\{\mathrm{rank}\, A | A \in S_+(G)\}.$$

In this paper we identify all graphs in $S(F, G)$ for which $\mathrm{mr}(F, G) \leq 2$ for any infinite field F (or, equivalently, the nullity is greater than or equal to $n-2$). We first show that the complements of such graphs have a simple explicit form (Theorems 1 and 2). We also give forbidden subgraph characterizations of this class of graphs in Theorems 6 and 7 and in Theorems 9 and 10 (the connected case). In the final section of the paper we depart from the theme of symmetric matrices and determine similar criteria for a Hermitian matrix to have minimum rank at most 2.

Before proceeding, we introduce some notation from graph theory.

Definition Given a graph $G = (V, E)$, the *complement* of G is the graph $G^c = (V, E^c)$. Given two graphs $G = (V_1, E_1)$ and $H = (V_2, E_2)$, with V_1 and V_2 disjoint, the *union* of G and H is $G \cup H = (V_1 \cup V_2, E_1 \cup E_2)$. The *join*, $G \vee H$, is the graph obtained from $G \cup H$ by adding an edge from each vertex of G to each vertex of H. If $S \subset V$, $G[S]$ denotes the subgraph of G induced by S. A vertex $u \in V$ is a *dominating vertex*, if u is adjacent to all other vertices in V.

Definition We denote the path on n vertices by P_n. The complete graph on n vertices will be denoted by K_n and we will refer to K_3 as the triangle. We abbreviate $K_n \cup \cdots \cup K_n$ (m times) to mK_n. The *complete multipartite graph*, $K_{m_1,m_2,\ldots,m_s}$ is the complement of $K_{m_1} \cup K_{m_2} \cup \cdots \cup K_{m_s}$. We will be particularly interested in the complete bipartite graphs $K_{m,n}$ as well as the complete tripartite graph $K_{3,3,3}$. In $K_{m,n}$ we may allow m or n to equal 0, in which case $K_{m,0} = mK_1$ or $K_{0,n} = nK_1$.

Definition A *clique* of a graph G is a set of pairwise adjacent vertices. A *maximal clique* is a clique that is not a proper subset of another clique. Given positive integers m, n, t with $t < \min\{m, n\}$, we will call the graph on $m + n - t$ vertices with exactly two maximal cliques, $\{1, 2, \ldots, m\}$, $\{m - t + 1, m - t + 2, \ldots, m - t + n\}$, the *clique sum* of K_m and K_n on K_t. We will also refer to such graphs as clique sums.

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Note that if $\mathrm{mr}(F,G) = 1$, then the rank 1 matrix A attaining this rank must have the form cxx^t, $c \in F^*$, $x \in F^n$. Let $W = \{i \in V(G) | x_i \neq 0\}$ and $m = |W|$. It follows that $G = K_m \cup K^c_{n-m}$. Moreover, if $m \geq 2$, any $A \in S(F, K_m \cup K^c_{n-m})$ has a nonzero entry and $J_m \oplus O_{n-n}$ (with J_m the m-by-m all ones matrix) is a matrix in $S(F, K_m \cup K^c_{n-m})$ with rank 1. Thus, we have

Observation 1 $\mathrm{mr}(F,G) = 1$ if and only if $G = K_m \cup K^c_{n-m}$, $m \geq 2$.

In particular, if G is connected, $\mathrm{mr}(F,G) = 1$ if and only if G is complete.

We note that one can reduce the problem of finding $\mathrm{mr}(F,G)$ to the connected case because it follows immediately from the definition that

Observation 2 If $G = \cup_{i=1}^k G_i$, then $\mathrm{mr}(F,G) = \sum_{i=1}^k \mathrm{mr}(F,G_i)$.

We now address the problem of finding all graphs for which $\mathrm{mr}(F,G) = 2$. We begin with some sufficient conditions.

Let $J_{m,n}$ be the m-by-n all ones matrix and let $J_n = J_{n,n}$. Let F_2 be the field with two elements.

Observation 3 If $m,\ n \geq 1$, $m + n \geq 3$, then $\mathrm{mr}(F, K_{m,n}) = 2$.
Apply Observation 1, and the fact that the matrix $\begin{bmatrix} 0 & J_{m,n} \\ J_{n,m} & 0 \end{bmatrix} \in S(F, K_{m,n})$ and has rank 2.

Observation 4 Let G be the clique sum of K_m and K_n on K_t.
(a) If $F \neq F_2$, then $\mathrm{mr}(F,G) = 2$,
(b) If $t = 1$, then $\mathrm{mr}(F,G) = 2$,
(c) If $m = n = t + 1$, then $\mathrm{mr}(F,G) = 2$,
(d) If $t \geq 2$, and $\max\{m,n\} \geq t + 2$, then $\mathrm{mr}(F_2,G) = 3$.

For (a) let α be distinct from 0 and -1. Then $\begin{bmatrix} J_m & 0 \\ 0 & 0 \end{bmatrix} + \alpha \begin{bmatrix} 0 & 0 \\ 0 & J_n \end{bmatrix} \in S(F,G)$ and has rank 2. (If $\mathrm{char}\, F \neq 2$, just take $\alpha = 1$.)

For (b) use the fact that $\begin{bmatrix} J_m & 0 \\ 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 \\ 0 & J_n \end{bmatrix} \in S(F,G)$ and has rank 2.

For (c) use the fact that $\begin{bmatrix} 0 & J_{1,t} & 0 \\ J_{t,1} & J_t & J_{t,1} \\ 0 & J_{1,t} & 0 \end{bmatrix} \in S(F,G)$ and has rank 2.

We will make no use of (d), so only give an abbreviated argument. The matrix $K = \begin{bmatrix} J_{m-t} & J_{m-t,t} & 0 \\ J_{t,m-t} & J_t & J_{t,n-t} \\ 0 & J_{n-t,t} & J_{n-t} \end{bmatrix} \in S(F_2, G)$ and has rank 3 ($\begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 1 \\ 0 & 1 & 1 \end{bmatrix}$ is a submatrix), so $\mathrm{mr}(F_2,G) \leq 3$.

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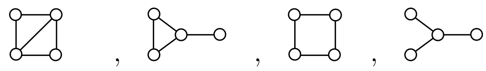
Now suppose that $M = \begin{bmatrix} A & J_{m-t,t} & 0 \\ J_{t,m-t} & B & J_{t,n-t} \\ 0 & J_{n-t,t} & C \end{bmatrix} \in S(F_2, G)$; so each non-diagonal entry of A, B, C is 1. Without loss of generality, say $n \geq t+2$. If some diagonal entry of C is 0, either $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 0 & 1 \end{bmatrix}$ or $\begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$ is a submatrix of M and rank $M \geq 3$. So assume that $C = J_{n-t}$. A similar argument shows that rank $M \geq 3$ if some diagonal entry of B is 0, so we assume $B = J_t$. Then it is obvious that if a diagonal entry of A is 0, that rank $M \geq 3$. And if $A = J_{m-t}$, then $M = K$ and rank $M \geq 3$. Consequently, $\mathrm{mr}(F_2, G) = 3$.

It follows from Observations 4(c), 4(b) and 3 that the graphs



diamond, paw, C_4, claw

on 4 vertices all have minimum rank 2 in any field.

On the other hand, easy necessary conditions follow from

Observation 5 If H is an induced subgraph of G, $\mathrm{mr}(F, G) \geq \mathrm{mr}(F, H)$.

For if B is a principal submatrix of A, rank $B \leq$ rank A.

It follows from Observation 5 that if $\mathrm{mr}(F, H) = 3$, H may not occur as an induced subgraph of any graph whose minimum rank is 2. In other words, H is a forbidden subgraph for the class of minimum rank 2 graphs. It is common to call such graphs H-*free*. Moreover, if $\mathcal{F}$ is a set of graphs, a graph is $\mathcal{F}$-free if it is H-free for each $H \in \mathcal{F}$.

2. Forbidden Subgraphs. We identify 6 forbidden subgraphs for the minimum rank 2 graphs.

1. P_4, which we label ①——②——③——④.

If $A \in S(F, P_4)$,

$$A = \begin{bmatrix} a_1 & b_1 & 0 & 0 \\ b_1 & a_2 & b_2 & 0 \\ 0 & b_2 & a_3 & b_3 \\ 0 & 0 & b_3 & a_4 \end{bmatrix} \quad \text{with } b_1 b_2 b_3 \neq 0.$$

Then rank $A \geq$ rank $\begin{bmatrix} b_1 & 0 & 0 \\ a_2 & b_2 & 0 \\ b_2 & a_3 & b_3 \end{bmatrix} = 3$. Since

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 1 & 1 \end{bmatrix} \in S(F, P_4), \mathrm{mr}(F, P_4) = 3.$$

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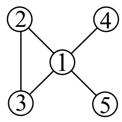
We note that any proper induced subgraph of P_4 has minimum rank ≤ 2.

2. Let G be the graph



which we will denote by $\ltimes$. Then if $A \in S(F, \ltimes)$,

$$A = \begin{bmatrix} d_1 & a_{12} & a_{13} & a_{14} & a_{15} \\ a_{12} & d_2 & a_{23} & 0 & 0 \\ a_{13} & a_{23} & d_3 & 0 & 0 \\ a_{14} & 0 & 0 & d_4 & 0 \\ a_{15} & 0 & 0 & 0 & d_5 \end{bmatrix}$$

with $a_{12}a_{13}a_{14}a_{15}a_{23} \neq 0$. Then

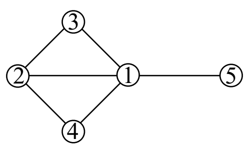
$$\text{rank } A \geq \text{ rank } A[1\ 2\ 4 \mid 1\ 3\ 5] = \text{rank} \begin{bmatrix} d_1 & a_{13} & a_{15} \\ a_{12} & a_{23} & 0 \\ a_{14} & 0 & 0 \end{bmatrix} = 3.$$

Since

$$\begin{bmatrix} 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \in S(F, \ltimes),$$

$\text{mr}(F, \ltimes) = 3$ for any field. Again, any proper induced subgraph of $\ltimes$ has minimum rank ≤ 2.

3. Let G be *the dart*



Following a similar argument to $\ltimes$, we see that $\text{mr}(F, \text{dart}) \geq \text{rank } A[1\ 2\ 4 \mid 1\ 3\ 5]$ $= 3$. Furthermore, the matrix $\begin{bmatrix} 1+1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \end{bmatrix} \in S\ (F,\ \text{dart})$ and has rank 3.

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So $\mathrm{mr}(F, \mathrm{dart}) = 3$ for any field. Again, any proper induced subgraph of the dart has minimum rank ≤ 2.

We also need to consider two disconnected graphs.

4. $G = P_3 \cup K_2$. By Observation 2,

$$\mathrm{mr}(F, P_3 \cup K_2) = \mathrm{mr}(F, P_3) + \mathrm{mr}(F, K_2).$$

Clearly, we have $\mathrm{mr}(F, P_3) = 2$ and $\mathrm{mr}(F, K_2) = 1$, so $\mathrm{mr}(F, P_3 \cup K_2) = 3$.

5. $G = 3K_2$. Then $\mathrm{mr}(F, 3K_2) = 3\mathrm{mr}(F, K_2) = 3$.

Our final graph is

6. $G = K_{3,3,3}$. Assume that the tripartite sets are $P_1 = \{1, 2, 3\}, P_2 = \{4, 5, 6\}$ and $P_3 = \{7, 8, 9\}$. Then any $A \in S(F, K_{3,3,3})$ has the form

$$A = \begin{bmatrix} d_1 & 0 & 0 & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} & a_{19} \\ 0 & d_2 & 0 & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} & a_{29} \\ 0 & 0 & d_3 & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} & a_{39} \\ a_{14} & a_{24} & a_{34} & d_4 & 0 & 0 & a_{47} & a_{48} & a_{49} \\ a_{15} & a_{25} & a_{35} & 0 & d_5 & 0 & a_{57} & a_{58} & a_{59} \\ a_{16} & a_{26} & a_{36} & 0 & 0 & d_6 & a_{67} & a_{68} & a_{69} \\ a_{17} & a_{27} & a_{37} & a_{47} & a_{57} & a_{67} & d_7 & 0 & 0 \\ a_{18} & a_{28} & a_{38} & a_{48} & a_{58} & a_{68} & 0 & d_8 & 0 \\ a_{19} & a_{29} & a_{39} & a_{49} & a_{59} & a_{69} & 0 & 0 & d_9 \end{bmatrix}$$

with $a_{ij} \neq 0$, for all i, j in distinct P_k's. We first note that if all the d_i's in any of the 3 diagonal blocks $\begin{bmatrix} d_r & 0 & 0 \\ 0 & d_s & 0 \\ 0 & 0 & d_t \end{bmatrix}$ are nonzero, then rank $A \geq 3$. So suppose each of these diagonal blocks has at least one d_i equal to 0. Then A has a *principal submatrix* of the form

$$\begin{bmatrix} 0 & a & c \\ a & 0 & b \\ c & b & 0 \end{bmatrix}, \quad a, b, c \neq 0.$$

Since the determinant is $2abc$, this is invertible if $\mathrm{char}\, F \neq 2$ and we conclude again that rank $A \geq 3$. Now consider the matrix

$$B = \begin{bmatrix} 0_3 & J_3 & J_3 \\ J_3 & 0_3 & J_3 \\ J_3 & J_3 & 0_3 \end{bmatrix} \in S(F, K_{3,3,3}).$$

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Clearly, rank $B \leq 3$. If char $F \neq 2$,

$$\text{rank } B \geq \text{rank} \begin{bmatrix} 0 & 1 & 1 \\ 1 & 0 & 1 \\ 1 & 1 & 0 \end{bmatrix} = 3$$

while if char $F = 2$, rank $B = 2$. We summarize the results of this section.

Observation 6 Let F be a field.

(a) $\text{mr}(F, P_4) = \text{mr}(F, \ltimes) = \text{mr}(F, \text{dart}) = \text{mr}(F, P_3 \cup K_2) = \text{mr}(F, 3K_2) = 3$.

(b) $\text{mr}(F, K_{3,3,3}) = \begin{cases} 3 & \text{if char } F \neq 2, \\ 2 & \text{if char } F = 2. \end{cases}$

Consequently, if G is a graph with $\text{mr}(F, G) \leq 2$, none of the graphs $P_4, \ltimes, \text{dart}, P_3 \cup K_2, 3K_2$ is an induced subgraph of G. We call such graphs $(P_4, \ltimes, \text{dart}, P_3 \cup K_2, 3K_2)$-free. If, in addition, char $F \neq 2$, $K_{3,3,3}$ is also not an induced subgraph of G. We will later see (Theorem 6) that this is a complete list of forbidden subgraphs for the class of minimum rank 2 graphs when char $F \neq 2$ and F is infinite. We shall also see that if char $F = 2$, then $K_{3,3,3}$ is replaced by $(P_3 \cup 2K_3)^c$.

3. Nondegenerate, bilinear symmetric forms on F^2. A *line* in F^2 is a one dimensional subspace of F^2. We let $S_2(F)$ denote the symmetric 2×2 matrices over F.

Let $B = [b_{ij}] \in S_2(F)$ be invertible. Then B defines a nondegenerate, bilinear symmetric form

$$(x, y) \mapsto x^t B y, \quad x, y \in F^2.$$

Given any line L in F^2 define its orthogonal complement (relative to the given form) by

$$L^\perp = \{y \in F^2 |\, y^t B x = 0 \quad \forall\, x \in L\}.$$

Since B is invertible, $L^\perp$ is a line in F^2.

Observation 7

(i) $y \in L^\perp \Leftrightarrow y^t B u = 0$ where $\{u\}$ is a basis of L.

(ii) $(L^\perp)^\perp = L$.

It is possible that $L^\perp = L$. In that case we say L is an *isotropic* line.

Now we consider the existence and the number of isotropic lines. For this purpose we can replace $B \in S_2(F)$ by any matrix congruent to it. We distinguish two cases.

Case (I) char $F \neq 2$.

Here we can assume B is a diagonal matrix, and since we may replace B by $\frac{1}{\alpha} B$ for any $\alpha \in F^*$, we can assume $B = \text{diag}(1, d), d \in F^*$. A line $L = \text{Span}\{x\}$ is isotropic if and only if

$$x_1^2 + d x_2^2 = 0. \tag{3.1}$$

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Thus, there exist isotropic lines iff $-d = \beta^2$ for some $\beta \in F^*$. If this is the case, we have exactly two distinct isotropic lines, $Sp\{\begin{bmatrix} x_1 \\ x_2 \end{bmatrix}\}$ and $Sp\{\begin{bmatrix} x_1 \\ -x_2 \end{bmatrix}\}$, where $x_1, x_2 \in F^*$ and satisfy (3.1). Otherwise, there are no isotropic lines.

We note that for the case $F = \mathbb{R}$, isotropic lines exist if and only if B has exactly one negative and one positive eigenvalue. So if B is (positive or negative) definite, there are no isotropic lines.

Case (II) $\operatorname{char} F = 2$.

If $b_{11} \neq 0$ or $b_{22} \neq 0$, we can diagonalize B by a congruence, and, as in the previous case, $L = \operatorname{span}\{x\}$ is isotropic if and only if equation (3.1) holds. If $-d$ is not a square, there is no isotropic line, and if $-d = \beta^2$, $x_1^2 - \beta^2 x_2^2 = (x_1 - \beta x_2)^2$, and there is *one* isotropic line, $x_1 - \beta x_2 = 0$. If $b_{11} = b_{22} = 0$, we can assume without loss of generality

$$B = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

so $x^t B y = x_2 y_1 + x_1 y_2 = x_2 y_1 - x_1 y_2$. Hence *every line L is isotropic.*

4. Graphs G with $\mathrm{mr}(F, G) \leq 2$. We need the following elementary result and include its proof for the sake of completeness.

Lemma 1 *Let $A \in S_n(F)$ with rank two. Then there is an invertible $B \in S_2(F)$ such that $A = U^t B U$, where U is a $2 \times n$ matrix.*

Proof: We may assume without loss of generality that the first two columns of A are linearly independent. Since A has rank two, there is a $2 \times n$ matrix W such that the matrix

$$A \begin{bmatrix} I_2 & W \\ 0 & I_{n-2} \end{bmatrix}$$

has columns $3, 4, \ldots, n$ equal to 0. Then

$$\begin{bmatrix} I_2 & W \\ 0 & I_{n-2} \end{bmatrix}^t A \begin{bmatrix} I_2 & W \\ 0 & I_{n-2} \end{bmatrix} = \begin{bmatrix} B & 0 \\ 0 & 0 \end{bmatrix}$$

with $B \in S_2(F)$ and invertible. Let $U = \begin{bmatrix} I_2 & -W \end{bmatrix}$. Then $A = U^t B U$. □

Theorem 1 *Let F be a field and G a graph on n vertices for which* $\mathrm{mr}(F, G) \leq 2$.

1. If $\operatorname{char} F \neq 2$, *then G^c is of the form*

$$(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r, \tag{4.1}$$

for appropriate nonnegative integers $s_1, s_2, k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.

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2. If $\operatorname{char} F = 2$*, then* G^c *is either of the form*

$$(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}) \vee K_r \tag{4.2}$$

or of the form

$$(K_{s_1} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r \tag{4.3}$$

for some appropriate nonnegative integers $k, s_1, s_2, \ldots, s_k, r, p_1, q_1, p_2, q_2, \ldots, p_k, q_k$ *with* $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.

Proof: The theorem is true if $\operatorname{mr}(F, G) = 1$ by Observation 1, so assume $\operatorname{mr}(F, G) = 2$, and let $A \in S(F, G)$ with rank $A = 2$. By Lemma 1, $A = U^t BU$, where B is an invertible 2×2 symmetric matrix over F and U is $2 \times n$. For $i = 1, 2, \ldots, n$, let $w_i = \begin{bmatrix} x_i \\ y_i \end{bmatrix}$ denote the ith column of U (so $[x_i, y_i]$ is the ith row of U^t), and let

$$L_i = Sp\{w_i\}.$$

Consider now $1 \le i \ne j \le n$. We have

$$ij \in E(G^c) \Leftrightarrow ij \notin E(G) \Leftrightarrow w_i^t B w_j = 0. \tag{4.4}$$

Suppose that r of the vectors $w_1, w_2, \ldots, w_n$ are 0; we may assume $w_{n-r+1} = w_{n-r+2} = \cdots = w_n = 0$. It follows that

$$G^c = H \vee K_r,$$

where H is the subgraph of G^c induced by $\{1, 2, \ldots, n-r\}$.

It remains to determine the structure of H. For $i = 1, 2, \ldots, n-r$, L_i is a line, and it follows from (4.4) that for $j = 1, 2, \ldots, n-r$,

$$ij \in E(G^c) \Leftrightarrow ij \notin E(G) \Leftrightarrow L_j = L_i^{\perp} \Leftrightarrow L_i = L_j^{\perp},$$

where the orthogonal complement is with respect to the bilinear form defined by B.

1. Suppose that $\operatorname{char} F \ne 2$. Then there are no isotropic lines or exactly two.

If there are two isotropic lines, denote them by $L^{(1)}$ and $L^{(2)}$. Consider now all $1 \le i \le n-r$ such that $L_i = L^{(1)}$. Suppose that there are s_1 such indices. Then H must have a connected component which is K_{s_1}. Similarly, working with $L^{(2)}$ we conclude $\exists\, s_2 \ge 0$ such that K_{s_2} is another connected component of H.

Now pick any line among $L_1, L_2, \ldots, L_{n-r}$ which is not isotropic, say L_z, and consider all $1 \le i \le n-r$ such that $L_i = L_z$. Suppose we have p_1 such indices. We now consider all $1 \le j \le n-r$ such that $L_j = L_i^{\perp}$. Suppose we have q_1 such indices. Then H must contain the component K_{p_1,q_1}. Repeating the process we complete the proof. (If there are no isotropic lines, $s_1 = s_2 = 0$.)

2. Suppose that $\operatorname{char} F = 2$. Then there is at most one isotropic line or each line is isotropic.

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Let us first assume that each line is isotropic. Then $ij \in E(H)$ if and only if $L_i = L_j$. Hence H is a disjoint union of complete graphs.

Assume now that there is at most one isotropic line, $L^{(1)}$. The vertices i with $L_i = L^{(1)}$ induce a connected component which is a complete graph. The remainder of the proof is identical to the last paragraph in part 1. □

Corollary 1 *Let G be a graph on n vertices. Then if $\mathrm{mr}_+(G) \le 2$, G^c is of the form*

$$(K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r \tag{4.5}$$

for appropriate nonnegative integers $k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.

Proof: The B in the proof of Theorem 1 is now positive definite, so there are no isotropic lines. Then the components K_{s_1} and K_{s_2} in (4.1) are absent. □

Theorem 2 *Let F be an infinite field.*

1. *If $\mathrm{char}\, F \neq 2$, let G be any graph whose complement is of the form (4.1).*
2. *If $\mathrm{char}\, F = 2$, let G be any graph whose complement is of the form (4.2) or (4.3).*

Then $\mathrm{mr}(F, G) \le 2$.

Proof: First, assume that $\mathrm{char}\, F \neq 2$ and let G^c have the form (4.1). Let $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It suffices to show there exists a $U \in F^{2,n}$ such that $A = U^t E U \in S(F, G)$. We note that the symmetric, bilinear form corresponding to E is given by

$$[x_1, x_2] E \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = x_1 y_2 + x_2 y_1.$$

Thus, the isotropic lines are $Sp \left\{ e_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\}$ and $Sp \left\{ e_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$.

Let $w_i, i = 1, 2, \ldots, n$ denote the columns of U. Choose $w_{n-r+1} = w_{n-r+2} = \cdots = w_n = 0$. Now let $w_1 = w_2 = \cdots = w_{s_1} = e_1$ and $w_{s_1+1} = w_{s_1+2} = \cdots = w_{s_1+s_2} = e_2$.

Now pick $a \in F^*$ and let $x^{(1)} = \begin{bmatrix} 1 \\ a \end{bmatrix}$. Then, for $L = Sp\{x^{(1)}\}$ we clearly have $L^\perp = Sp\{y^{(1)}\}$, where $y^{(1)} = \begin{bmatrix} 1 \\ -a \end{bmatrix}$. So among the remaining w_i's we pick p_1 to be equal to $x^{(1)}$ and q_1 to be equal to $y^{(1)}$. We now choose $b \in F^*$, $b \neq a$, $b \neq -a$ and pick p_2 (resp. q_2) of the remaining w_i's to be equal to $\begin{bmatrix} 1 \\ b \end{bmatrix}$ $\left(\text{resp } \begin{bmatrix} 1 \\ -b \end{bmatrix}\right)$. Since F is infinite and since the only zeros in the submatrix $A[\{1, 2, \ldots, n-r\}]$ result when x, y belong to the same isotropic line or belong to one of the pairs $L, L^\perp$ above, we can continue the process and obtain a matrix A of rank ≤ 2 in $S(F, G)$.

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We now assume that char $F = 2$. Consider first the case where the complement of G is of the form

$$(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}) \vee K_r.$$

Let $E = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}$. It suffices to show there exists a $U \in F^{2,n}$ such that $A = U^t EU \in S(F,G)$. Each line in F^2 is isotropic. Let $w_i, i = 1, 2, \ldots, n$ denote the columns of U. Choose $w_{n-r+1} = w_{n-r+2} = \cdots = w_n = 0$. Choose k distinct lines $L_1, L_2, \ldots, L_k$ in F^2 and choose $v_1, v_2, \ldots, v_k$ such that $L_i = Sp\{v_i\}$, $i = 1, 2, \ldots, k$. Let $w_1 = w_2 = \cdots = w_{s_1} = v_1$, $w_{s_1+1} = w_{s_1+2} = \cdots = w_{s_1+s_2} = v_2, \ldots, w_{n-r-s_k+1} = \cdots = w_{n-r} = v_k$. Then $U^t EU \in S(F,G)$.

We now assume that the complement of G is of the form

$$(K_{s_1} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r.$$

Let $E = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$. Notice that $Sp\{e := [1,1]^t\}$ is an isotropic line. Choose $w_{n-r+1} = w_{n-r+2} = \cdots = w_n = 0$. Choose $w_1, w_2, \ldots, w_{s_1} = e$.

Now pick $a \in F\backslash\{1\}$ and let $x^{(1)} = [1,a]^t$ and $y^{(1)} = [a,1]^t$. Then $Sp\{x^{(1)}\}$ is orthogonal to $Sp\{y^{(1)}\}$. Among $w_{s_1+1}, w_{s_1+2}, \ldots, w_{n-r}$ we pick the first p_1 equal to $x^{(1)}$ and the q_1 after these equal to $y^{(1)}$. Since F has an infinite number of elements, we can continue this process and obtain a matrix A of rank ≤ 2 in $S(F,G)$. □

Theorem 2 is false if F is finite.

Example Let F be a finite field with q elements and let $G^c = (q+2)K_2$. Suppose $\mathrm{mr}(F,G) = 2$ (it can't be 1), and let $A \in S(F,G)$ with rank $A = 2$. By Lemma 1, A can be factored $A = U^t BU$. Write $U = [w_1, w_2, \ldots, w_{2q+4}] \in F^{2,2q+4}$. If $0, 1, a_1, a_2, \ldots, a_{q-2}$ are the elements of F, there are $q+1$ distinct lines in F^2, those spanned by $\begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ a_1 \end{bmatrix}, \ldots, \begin{bmatrix} 1 \\ a_{q-2} \end{bmatrix}$. Consequently, there is a vertex i in one K_2 and a vertex j in another K_2 such that $w_j = \alpha w_i$, $\alpha \in F^*$. Since ij is an edge of G, w_i is not orthogonal to w_j, so it follows that w_i is not isotropic. Let k be the neighbor of i in G^c. Then $w_k \perp w_i$, so $w_k \perp w_j$ which implies that jk is an edge in G^c, a contradiction. Therefore $\mathrm{mr}(F,G) \geq 3$.

For the case $F = \mathbb{R}$ we have

Corollary 2 *If G is a graph whose complement is of the form* (4.1)*, then* $\mathrm{mr}(G) \leq 2$. *Furthermore, if* $\mathrm{mr}(G) = 2$*, the minimum is attained at a matrix with one positive and one negative eigenvalue.*

Proof: This follows from Theorem 2, Theorem 1, and Sylvester's law of inertia. □

Theorem 3 *Let G be a graph on n vertices. Then* $\mathrm{mr}_+(G) \leq 2$ *if and only if G^c has the form* (4.5).

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Proof: The forward implication is Corollary 1. For the reverse implication, instead of adapting the proof of Theorem 2, we give a geometrical argument. Let G be a graph of the form (4.5). Define a $2 \times n$ matrix $U = [u_1, u_2, \ldots, u_n]$ as follows. For each $i \in V(K_r)$, let $u_i = 0$. Choose nonzero vectors $w_1, w_2, \ldots, w_k \in \mathbb{R}^2$ such that for $i \neq j$, w_i and w_j are linearly independent and $w_i^t w_j \neq 0$. Let $x_1, x_2, \ldots, x_k$ be nonzero vectors such that $w_i^t x_i = 0$, $i = 1, 2, \ldots, k$. Let S_i, T_i be the color classes of $K_{p_i,q_i}, i = 1, 2, \ldots, k$. For $v \in S_i$, define $u_v = w_i$ and for $v \in T_i$ define $u_v = x_i$. Then $U^t U \in S_+(G)$ and has rank ≤ 2. □

5. Forbidden Subgraph Characterizations. We now show how the graphs in (4.1), those in (4.2) and (4.3), and those in (4.5) can be characterized in terms of forbidden subgraphs through a series of propositions. It is clear that a graph G is a union of complete graphs if and only if G is P_3-free. Taking complements, a graph G is a complete multipartite graph if and only if G is $(K_2 \cup K_1)$-free. It follows that G is a complete bipartite graph if and only if G is (triangle, $K_2 \cup K_1$)-free.

Proposition 1 *The graph G is a union of complete bipartite graphs if and only if G is* (triangle, P_4)*-free.*

Forward implication. Since G is bipartite, G is triangle-free. If P_4 is an induced subgraph of G, it is an induced subgraph of some component, and then so is $K_2 \cup K_1$, a contradiction.

Reverse implication. Let $H = (V, E)$ be a component of a (triangle, P_4)-free graph G. Since H is P_4-free, H has no odd cycles, and is therefore bipartite. If H is not complete bipartite, P_4 is an induced subgraph, contrary to assumption. □

Proposition 2 *Let $G = (V, E)$ be a connected* paw*-free graph containing a triangle. Then*

1. *every vertex of G lies on a triangle.*
2. *G does not contain a cut vertex.*

Proof: Suppose u is not on a triangle, let k be the minimum distance from u to a triangle, $G[\{w, x, y\}]$, and let $P = [u_0 = u, u_1, u_2, \ldots, u_k = w]$ be a path from u to the triangle. Since $G[\{u_{k-1}, w, x, y\}]$ is not a paw, we may assume $u_{k-1}x \in E$. Then $\{u_{k-1}, w, x\}$ induces a triangle at distance $k-1$ from u, a contradiction.

Now suppose w is a cut vertex of G, and assume $\{w, x, y\}$ induces a triangle. Then there is a vertex u adjacent to w and lying in a component of $G - w$ distinct from the component containing $\{x, y\}$. Then $\{u, w, x, y\}$ induces a paw, contrary to assumption. □

Proposition 3 *A connected* (paw, diamond)*-free graph that contains a triangle is complete.*

Proof: Let G be such a graph. The proof is by induction on n, the number of vertices of G. If $n = 3$, G is K_3. Suppose $n \geq 4$ and that the theorem is true for all graphs with fewer than n vertices. Assume $\{u_1, u_2, u_3\}$ induces a triangle and let u

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be a vertex of G distinct from u_1, u_2, u_3. By Proposition 2, $G - u$ is connected, so is complete by the induction hypothesis.

Since G is connected, u is adjacent to a vertex x. Let y, z be any other vertices of $G - u$. Since $G - u$ is complete, $\{x, y, z\}$ induces a triangle. Since $G[\{u, x, y, z\}]$ is neither a paw nor a diamond, it is K_4, and u is adjacent to y and z. Consequently, G is complete. □

Proposition 4 *A graph G can be expressed as the union of complete graphs and of complete bipartite graphs if and only if G is (P_4,* paw, diamond*)-free.*

Remark: The decomposition of a graph in this proposition is not unique. For example, $G = K_4 \cup K_3 \cup K_2 \cup K_2 \cup K_{2,2} \cup K_{1,5}$ can be thought of as the union of 4 complete graphs and 2 complete bipartite graphs, as the union of 2 complete graphs and 4 complete bipartite graphs, or of 3 of each.

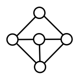
Forward implication. None of P_4, paw, and diamond can be an induced subgraph of a component of G.

Reverse implication. We may express

$$G = E_1 \cup \cdots \cup E_j \cup H_1 \cup \cdots \cup H_k$$

where the E_i and H_i are the components of G, each E_i contains a triangle, and each H_i is (triangle, P_4)-free. By Proposition 1, each H_i is a complete bipartite graph, and by Proposition 3, each E_i is complete. □

Our next step is to characterize the class of graphs obtained by taking the join of a graph in Proposition 4 and a complete graph. Let $\hat{W}_4$ be the graph on 5 vertices



Theorem 4 *A graph G has the form*

$$(5.1) \qquad (K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_t} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$$

for nonnegative integers $t, s_1, s_2, \ldots s_t, k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$, if and only if G is $(P_4, \text{paw} \cup K_1, \text{diamond} \cup K_1, \hat{W}_4, K_{2,2,2})$-free.

Forward implication. Any induced subgraph of (5.1) containing a vertex of K_r must contain a dominating vertex, so cannot be any of the graphs in

$$\mathcal{F} = \{P_4, \text{paw} \cup K_1, \text{diamond} \cup K_1, \hat{W}_4, K_{2,2,2}\}.$$

Therefore, if a graph in $\mathcal{F}$ is induced, it must be a subgraph of $K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_t} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}$. Since the diamond is an induced subgraph of both $\hat{W}_4$ and $K_{2,2,2}$, it follows from Proposition 4 that no graph in $\mathcal{F}$ can be an induced

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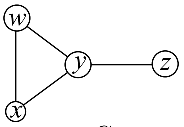
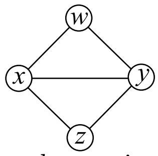
subgraph of $K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_t} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}$. Therefore, a graph of the form (5.1) must be $\mathcal{F}$-free.

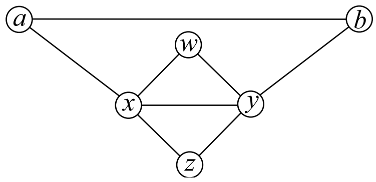
Reverse implication. Let $G = (V, E)$ be an $\mathcal{F}$-free graph. Let D be the set of dominating vertices of G, let $r = |D|$, and let $C = V \backslash D$. Then $G = G[C] \vee K_r$. We now show that $G[C]$ is a (paw, diamond)-free graph.

Suppose that 

is an induced subgraph of $G[C]$. Since $y \notin D$, there exists a vertex $u \in C$ not adjacent to y. Since paw $\cup K_1$ is not induced, u is adjacent to at least one of w, x, z. If u is adjacent to a nonempty proper subset of $\{w, x, z\}$, P_4 is induced in $G[C]$, and therefore G, contrary to hypothesis, while if u is adjacent to each of w, x, z, $\hat{W}_4$ is induced, also contrary to assumption. Therefore $G[C]$ is paw-free.

Suppose that 

is an induced subgraph of $G[C]$. Since x, y are not dominating vertices, there exist vertices $a, b \in C$ with a not adjacent to y and b not adjacent to x. If $a = b$, then since diamond $\cup K_1$ is not induced, a is adjacent to at least one of w and z, say w. Then $\{a, w, x, y\}$ induces a paw, a contradiction. So $a \neq b$. Then, since diamond $\cup K_1$ is not induced, a is adjacent to at least one of w, x, z and b is adjacent to at least one of w, y, z. Suppose that a and x are not adjacent. Then, if a is adjacent to only one of w and z, P_4 is induced, and if a is adjacent to w and z, $\hat{W}_4$ is induced, both contrary to assumption. Therefore a is adjacent to x, and by symmetry b is adjacent to y. Then a and b are adjacent, else $\{a, x, y, b\}$ induces P_4. So



is a subgraph of $G[C]$. Since $G[C]$ is paw-free, $G[\{a, w, x, y\}]$ has more than 4 edges. But a and y are not adjacent, so aw is an edge. A similar argument shows that bw, az, and bz are edges. Since ay, bx, wz are not edges, $G[\{a, b, w, x, y, z\}] = K_{2,2,2}$, contrary to assumption. Therefore $G[C]$ is diamond-free.

Because $G[C]$ is (paw,diamond)-free and P_4-free by hypothesis, by Proposition 4, $G[C]$ can be expressed as the union of complete graphs and complete bipartite graphs. Then $G = G[C] \vee K_r$ is of the form (5.1). □

We can now give a forbidden subgraph characterization of the graphs in (4.1).

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Corollary 3 *A graph G has the form*

$$(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r \tag{5.2}$$

for nonnegative integers $s_1, s_2, k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$, if and only if G is $(P_4, \text{paw} \cup K_1, \text{diamond} \cup K_1, \hat{W}_4, K_{2,2,2}, 3K_3)$-free.

Note that the 6 forbidden graphs in this statement are the complements of the forbidden subgraphs in section 2.

Forward implication. By Theorem 4 it suffices to show that G is $3K_3$-free. Since $3K_3$ does not contain a dominating vertex, if it is an induced subgraph of G, it must be induced in $K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}$. But K_3 is not an induced subgraph of a union of complete bipartite graphs, so $3K_3$ must be induced in $K_{s_1} \cup K_{s_2}$, which is impossible.

Reverse implication. By Theorem 4, G has the form

$$(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_t} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r.$$

We may assume that $s_1, s_2, \ldots, s_t \geq 3$. Then if $t \geq 3$, $3K_3$ is induced. So $t \leq 2$ and G has the form (5.2). □

Taking complements in Corollary 3 we have

Theorem 5 *Let G be a graph. Then G^c has the form* (5.2) *if and only if G is $(P_4, \text{dart}, \ltimes, P_3 \cup K_2, 3K_2, K_{3,3,3})$-free.*

Combining Theorems 1, 2 ($\operatorname{char} F \neq 2$), and 5,

Theorem 6 *Let G be a graph and let F be an infinite field such that $\operatorname{char} F \neq 2$. Then the following are equivalent:*

1. $\operatorname{mr}(F, G) \leq 2$.
2. *G^c has the form $(K_{s_1} \cup K_{s_2} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for nonnegative integers $s_1, s_2, k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.*
3. *G is $(P_4, \ltimes, \text{dart}, P_3 \cup K_2, 3K_2, K_{3,3,3})$-free.*

In order to state analogous results when $\operatorname{char} F = 2$ we need

Corollary 4 *A graph G has the form*

$$(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}) \vee K_r \tag{5.3}$$

or has the form

$$(K_{s_1} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r \tag{5.4}$$

for some appropriate nonnegative integers $k, s_1, s_2, \ldots, s_k, r, p_1, q_1, p_2, q_2, \ldots, p_k, q_k$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$, if and only if G is $(P_4, \text{paw} \cup K_1$, diamond $\cup K_1$, $\hat{W}_4, K_{2,2,2}, P_3 \cup 2K_3)$-free.

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Forward implication. It suffices by Theorem 4 to show that $P_3 \cup 2K_3$ is not induced. Suppose that it is. If G is of the form (5.3), then since $P_3 \cup 2K_3$ does not have a dominating vertex, it must be induced in $K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}$ implying that P_3 is induced in a complete graph, a contradiction. If G is of the form (5.4), $P_3 \cup 2K_3$ must be induced in $K_{s_1} \cup K_{p_1,q_1} \cup K_{q_2,q_2} \cup \cdots \cup K_{p_k,q_k}$, implying that $2K_3$ is induced in K_{s_1}, a second contradiction. Thus $P_3 \cup 2K_3$ is not induced.

Reverse implication. From Theorem 4 we see that G has the form

$$(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_t} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r.$$

If $K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_t} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}$ is P_3-free, it is the union of complete graphs and G has the form (5.3). So assume that P_3 is induced in K_{p_1,q_1}. Since $P_3 \cup 2K_3$ is not induced in G, $s_i \geq 3$ for at most one i. Then the remaining K_{s_i} are complete bipartite graphs and G has the form (5.4). □

Combining Theorems 1, 2 ($\operatorname{char} F = 2$), and taking the complements of the graphs in Corollary 4 we have

Theorem 7 *Let G be a graph and let F be an infinite field such that* $\operatorname{char} F = 2$. *Then the following are equivalent:*

1. $\operatorname{mr}(F, G) \leq 2$.

2. G^c *is either of the form*

$$(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_k}) \vee K_r \tag{5.5}$$

or of the form

$$(K_{s_1} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r \tag{5.6}$$

for some appropriate nonnegative integers $k, s_1, s_2, \ldots, s_k, r, p_1, q_1, p_2, q_2, \ldots, p_k, q_k$ *with* $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.

3. G *is* $(P_4, \ltimes, \text{dart}, P_3 \cup K_2, 3K_2, (P_3 \cup 2K_3)^c)$*-free.*

Theorems 6 and 7 are fairly definitive answers to the problem posed at the outset of the paper. The problem of characterizing which graphs have $\operatorname{mr}(F, G) \leq 2$ for finite fields is intricate and will be presented in a subsequent paper. The second criterion of these theorems is undoubtedly the characterization that should be used to algorithmically determine whether or not a graph has minimum rank less than or equal to 2. The third criterion gives insight into the obstructions that prevent a graph from having rank less than 3 by establishing that the 6 forbidden graphs in section 2 comprise a complete list of minimal rank 3 graphs for an infinite field with $\operatorname{char} F \neq 2$ and the same list with one substitution is a complete list of minimal rank 3 graphs when $\operatorname{char} F = 2$. The equivalence of 2 and 3 in each theorem is of some interest graph theoretically as there is no transparent connection between them. Problems of finding forbidden subgraphs for graph classes obtained through unions and joins is investigated systematically in [1]. The problem of characterizing graphs with $\operatorname{mr}(F, G) \leq k$ for $k \geq 3$ appears very difficult.

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We can also give a forbidden subgraph characterization for the class of graphs (4.5).

Corollary 5 *A graph G has the form*

$$(K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r \tag{5.7}$$

for nonnegative integers $k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0, i = 1, 2, \ldots, k$, if and only if G is $(P_4, K_3 \cup K_1, \hat{W}_4, K_{2,2,2})$-free.

Forward implication. Suppose $K_3 \cup K_1$ were an induced subgraph. Since it has no dominating vertex, it would be induced in the union of complete bipartite graphs, which is impossible. Moreover, $P_4, \hat{W}_4$, and $K_{2,2,2}$ are all forbidden by Theorem 4.

Reverse implication. Since G is $(K_3 \cup K_1)$-free, it is also (paw $\cup K_1$, diamond $\cup K_1$)-free. Then by Theorem 4, G has the form

$$(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_t} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r.$$

If $s_1 \geq 3$ and $k \geq 1$, then $K_3 \cup K_1$ is an induced subgraph, contrary to assumption. Suppose for some $i = 1, \ldots, t$ that $s_i \geq 3$, so that $k = 0$. If a second s_i is positive, $K_3 \cup K_1$ is again an induced subgraph. So all but one s_i is 0 and G is complete which is trivially of the form (5.7). So we may assume $s_i \leq 2$ for each i. But then all of the K_{s_i} are bipartite and again G has the form (5.7). □

Combining Theorem 3 and Corollary 5 we have

Theorem 8 *Given a graph G, the following are equivalent.*

1. $\mathrm{mr}_+(G) \leq 2$.
2. *G^c has the form $(K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for nonnegative integers $k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.*
3. *G is $(P_4, \mathrm{claw}, P_3 \cup K_2, 3K_2)$-free.*

We conclude this section by specializing our results to the case that G is connected. Our first result is of some interest graph theoretically.

Proposition 5 *Let G be connected. Then G is $(P_4, \mathrm{dart}, \ltimes)$-free if and only if G is $(P_4, P_3 \cup K_1, K_2 \cup 2K_1)$-free.*

Reverse implication. If the dart is an induced subgraph, so is $P_3 \cup K_1$ and if $\ltimes$ is an induced subgraph, so is $K_2 \cup 2K_1$.

Forward implication. Suppose that $P_3 \cup K_1 =$ (x)—(y)—(z) (w) is an induced subgraph of G. Let k be the minimum distance from w to (x)—(y)—(z). Then $k \geq 2$, but if $k > 2$, P_4 is induced. Therefore $k = 2$. Let u be the intermediate vertex on a shortest path from w to (x)—(y)—(z), so that u is adjacent to w and at least one of x, y, z. If u is adjacent to a proper subset of $\{x, y, z\}$, P_4 is induced, and if u is adjacent to each of x, y, z, $G[\{x, y, z, u, w\}]$ is a dart. So G is $(P_3 \cup K_1)$-free.

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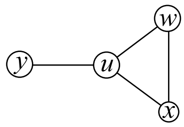
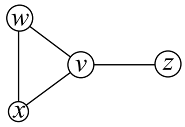
Suppose that $K_2 \cup 2K_1 =$ (w)—(x) (y) (z) is induced. If the distance from y to (w)—(x) is greater than 2, P_4 is induced, so the distance is 2. Let u be the intermediate vertex on a shortest path from y to (w)—(x). If u is not adjacent to both w and x, P_4 is induced, so assume u is adjacent to both. Then



is an induced subgraph of G. If z is adjacent to u, then $G[w, x, u, y, z]$ is $\ltimes$, a contradiction. So z and u are not adjacent. Replacing y by z in the argument with y and (w)—(x), there is a vertex $v \neq u$ such that



is an induced subgraph of G. If u and v are not adjacent, $G[u, w, v, z]$ is P_4, so u and v are adjacent. If v and y are not adjacent, $G[y, u, v, z]$ is P_4, so v and y are adjacent. Then $G[w, x, v, y, z]$ is $\ltimes$, a contradiction. So $K_2 \cup 2K_1$ is not induced. □

If char $F \neq 2$, then from Theorem 6 we see that G is connected only if $r = 0$. We now characterize this class of graphs.

Proposition 6 *A graph can be expressed as the union of at most two complete graphs and of complete bipartite graphs if and only if G is (P_4,* paw, diamond, *$3K_3$)-free.*

Forward implication. G is (P_4, paw, diamond)-free by Proposition 4, and $3K_3$ is not induced since K_3 is not induced in a bipartite graph.

Reverse implication. By Proposition 4,

$$G = K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_t} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k},$$

where we may assume $s_1, s_2, \ldots, s_t \geq 3$. Since $3K_3$ is not induced, $t \leq 2$. □

Taking complements we have

Corollary 6 *G^c can be expressed as the union of at most two complete graphs and of complete bipartite graphs if and only if G is $(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3})$-free.*

We tighten this up by using the fact that G is connected.

Corollary 7 *Let G be connected. Then G is $(P_4, \text{dart}, \ltimes, K_{3,3,3})$-free if and only if G is $(P_4, P_3 \cup K_1, K_2 \cup 2K_1, K_{3,3,3})$-free.*

Proof: This is an immediate consequence of Proposition 5. □

Combining Corollaries 6 and 7, we have

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Corollary 8 *If G is a connected graph, then G^c can be expressed as the union of at most two complete graphs and of complete bipartite graphs if and only if G is* $(P_4, \text{dart}, \ltimes, K_{3,3,3})$*-free.*

Therefore, in the connected case, Theorem 6 becomes

Theorem 9 *Let G be a connected graph and let F be an infinite field with* $\operatorname{char} F \neq 2$. *Then the following are equivalent:*

1. $\operatorname{mr}(F, G) \leq 2$.
2. *G^c can be expressed as the union of at most 2 complete graphs and of complete bipartite graphs.*
3. *G is* $(P_4, \text{dart}, \ltimes, K_{3,3,3})$*-free.*

We can obtain analogues of Proposition 6 and Corollary 6 for the case $\operatorname{char} F = 2$.

Proposition 7 *A graph can be expressed as the union of complete graphs or as the union of at most one complete graph and of complete bipartite graphs if and only if G is* $(P_4, \text{paw}, \text{diamond}, P_3 \cup 2K_3)$*-free.*

Forward implication. G is (P_4, paw, diamond)-free by Proposition 4 and it is easy to check that $P_3 \cup 2K_3$ is not induced.

Reverse implication. If G is P_3-free, then G is a union of complete graphs. So we may assume that G has P_3 as an induced subgraph. By Proposition 4, G is the union of complete graphs and of complete bipartite graphs with P_3 induced in one of the complete bipartite graphs. Then $2K_3$ cannot be induced, which means that at most one of the complete graphs has 3 vertices. Then G can be expressed as the union of at most one complete graph and of complete bipartite graphs. □

Taking complements we have

Corollary 9 *G^c can be expressed as the union of complete graphs or as the union of at most one complete graph and of complete bipartite graphs if and only if G is* $(P_4, P_3 \cup K_1, K_2 \cup 2K_1, (P_3 \cup 2K_3)^c)$*-free.*

Proposition 8 *Let G be connected. Then G is* $(P_4, \text{dart}, \ltimes, (P_3 \cup 2K_3)^c)$*-free if and only if G is* $(P_4, P_3 \cup K_1, K_2 \cup 2K_1, (P_3 \cup 2K_3)^c)$*-free.*

Proof: This follows immediately from Proposition 5. □

Combining with Corollary 9, we have

Corollary 10 *If G is a connected graph, then G^c can be expressed as the union of complete graphs or as the union of at most one complete graph and of complete bipartite graphs if and only if G is* $(P_4, \text{dart}, \ltimes, (P_3 \cup 2K_3)^c)$*-free.*

In the connected case we must have $r = 0$ in Theorem 7. So we have by Corollary 10

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Theorem 10 *Let G be a connected graph and let F be an infinite field with* $\operatorname{char} F = 2$*. Then the following are equivalent:*

1. $\operatorname{mr}(F, G) \leq 2$.
2. G^c *can be expressed as the union of complete graphs or as the union of at most one complete graph and of complete bipartite graphs.*
3. G *is* $(P_4, \text{dart}, \ltimes, (P_3 \cup 2K_3)^c)$*-free.*

Finally, G is connected in Theorem 8 only if $r = 0$. Making use of Proposition 1, we have

Theorem 11 *Given a connected graph G, the following are equivalent.*

1. $\operatorname{mr}_+(G) \leq 2$.
2. G^c *is the union of complete bipartite graphs.*
3. G *is* (K_3^c, P_4)*-free. [G is P_4-free with independence number ≤ 2.]*

6. Minimum Hermitian rank. In the previous sections we characterized for any infinite field F those graphs G such that $\operatorname{mr}(F, G) \leq 2$. This included the case $F = \mathbb{C}$. But besides symmetric matrices with entries in $\mathbb{C}$, we could also study the case of Hermitian matrices. Given a graph $G = (V, E)$ with vertex set $V = \{1, 2, \ldots, n\}$, let $H(G)$ be the set of all Hermitian $n \times n$ matrices $A = [a_{ij}]$ such that $a_{ij} \neq 0$, $i \neq j$, if and only if $ij \in E$. There is no restriction on the diagonal entries of A. We define

$$\operatorname{hmr}(G) = \min\{\operatorname{rank} A \mid A \in H(G)\}.$$

Inspection of the six graphs presented in Section 2 shows that all of them except $K_{3,3,3}$ have $\operatorname{hmr}(G) = 3$, and that $\operatorname{hmr}(K_{3,3,3}) = 2$. In this section we characterize those graphs G with $\operatorname{hmr}(G) \leq 2$. We will see that the forbidden subgraphs are the first five graphs presented in Section 2.

A *line* in $\mathbb{C}^2$ is a one dimensional subspace of $\mathbb{C}^2$ over $\mathbb{C}$. Let $B = [b_{ij}]$ be any invertible Hermitian 2×2 matrix. Given any line L in $\mathbb{C}^2$ define its orthogonal complement (relative to B) by

$$L^\perp = \{y \in \mathbb{C}^2 \mid y^* B x = 0 \quad \forall x \in L\}.$$

Observation 7 (with y^t replaced by y^*) is also valid for this case. We call a line L *isotropic* if $L^\perp = L$.

We now consider the existence and the number of isotropic lines. For this we may replace B by any matrix of the form $S^* B S$, where S is a 2×2 invertible matrix over $\mathbb{C}$; we may also multiply it by -1. Therefore, we can assume that either $B = \operatorname{diag}(1, -1)$ or $B = \operatorname{diag}(1, 1)$. Let $[x_1, x_2]^t$ be a nonzero vector in L. Then L is isotropic if and only if $x^* B x = 0$. Thus, there exist isotropic lines if and only if $B = \operatorname{diag}(1, -1)$ and $|x_1|^2 - |x_2|^2 = 0$. If this is the case, we can have $x_1 \neq 0$ and $x_2 \neq 0$; we may take $x_1 = 1$, and then L is an isotropic line if and only if $|x_2|^2 = 1$. In particular, there can be infinitely many isotropic lines.

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Theorem 12 *Let G be a graph on n vertices. Then, if* $\mathrm{hmr}(G) \le 2$*, G^c is of the form*

$$(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_t} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r \tag{6.1}$$

for nonnegative integers $t, s_1, s_2, \ldots s_t, k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.

The proof of this theorem is similar to the proof of Theorem 1, except that in this case there are an infinite number of isotropic lines if there are isotropic lines.

Theorem 13 *Let G be any graph whose complement is of the form* (6.1)*. Then* $\mathrm{hmr}(G) \le 2$.

Proof: Let $E = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$. It suffices to show there exists a $U \in \mathbb{C}^{2,n}$ such that $A = U^*EU \in H(G)$. Let w_i, $i = 1, 2, \ldots, n$ denote the columns of U, and let $e_\zeta = \begin{bmatrix} 1 \\ \zeta \end{bmatrix}$. If $\zeta \in \mathbb{C}$ and $|\zeta| = 1$, the line spanned by e_ζ is isotropic. Choose t distinct complex numbers $\zeta_1, \zeta_2, \ldots, \zeta_t$ with $|\zeta_i| = 1$ for $i = 1, 2, \ldots, t$. Choose $w_{n-r+1} = w_{n-r+2} = \cdots = w_n = 0$. For $i = 1, 2, \ldots, t$, let $f_i = 1 + \sum_{j=1}^{i-1} s_j$ (with the understanding that $f_1 = 1$), let $l_i = \sum_{j=1}^{i} s_j$, and choose $w_{f_i}, w_{f_i+1}, \ldots, w_{l_i} = e_{\zeta_i}$.

Now pick a nonzero $\alpha_1 \in \mathbb{C}$ with $|\alpha_1| \ne 1$. Then $x^{(1)} = [1, \alpha_1]^t \in \mathbb{C}^2$ is a non-isotropic vector and $y^{(1)} = [\overline{\alpha_1}, 1]^t$ is a vector in $\mathbb{C}^2$ orthogonal to $x^{(1)}$. Among the remaining w_i's we pick the first p_1 equal to $x^{(1)}$ and the q_1 after these equal to $y^{(1)}$. Since $\mathbb{C} \setminus \{\zeta \mid |\zeta| = 1\}$ has an infinite number of elements, we can continue this process and obtain a matrix A of rank ≤ 2 in $H(G)$. □

Corollary 11 *If* $\mathrm{hmr}(G) = 2$*, the minimum is attained at a Hermitian matrix with one positive and one negative eigenvalue.*

Combining Theorems 12, 13 and taking complements in Theorem 4 gives

Theorem 14 *Let G be a graph. Then the following are equivalent:*

1. $\mathrm{hmr}(G) \le 2$.
2. *G^c has the form $(K_{s_1} \cup K_{s_2} \cup \cdots \cup K_{s_t} \cup K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for nonnegative integers $t, s_1, s_2, \ldots, s_t, k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.*
3. *G is $(P_4, \ltimes, \mathrm{dart}, P_3 \cup K_2, 3K_2)$-free.*

Let $H_+(G)$ be the set of positive semidefinite Hermitian matrices in $H(G)$ and let

$$\mathrm{hmr}_+(G) = \min\{\mathrm{rank}\, A \mid A \in H_+(G)\}.$$

Theorem 15 *Let G be a graph. Then the following are equivalent:*

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1. $\mathrm{hmr}_+(G) \le 2$.
2. G^c *has the form* $(K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ *for nonnegative integers* $k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ *with* $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.
3. G *is* $(P_4, \mathrm{claw}, P_3 \cup K_2, 3K_2)$*-free.*

Proof: As in the proof of Corollary 1, there are no isotropic lines, so $K_{s_1}, K_{s_2}, \ldots, K_{s_k}$ in equation (6.1) are absent. Thus (1) $\Longrightarrow$ (2). That (2) $\Longrightarrow$ (1) and (2) $\Longleftrightarrow$ (3) follows from Theorem 8. □

Recall the definition of $\tau(G)$ from the introduction. Theorem 15 says that $\tau(G) \ge n-2$ if and only if G^c has the form $(K_{p_1,q_1} \cup K_{p_2,q_2} \cup \cdots \cup K_{p_k,q_k}) \vee K_r$ for nonnegative integers $k, p_1, q_1, p_2, q_2, \ldots, p_k, q_k, r$ with $p_i + q_i > 0$, $i = 1, 2, \ldots, k$.

We now specialize the result of this section to the case that G is connected. As before, we see that G is connected only if $r = 0$. Taking complements in Proposition 4, we obtain

Corollary 12 G^c *can be expressed as the union of complete graphs and of complete bipartite graphs if and only if* G *is* $(P_4, P_3 \cup K_1, K_2 \cup 2K_1)$*-free.*

Applying Proposition 5, Theorem 14 for a connected graph becomes

Theorem 16 *Let* G *be a connected graph. Then the following are equivalent:*
1. $\mathrm{hmr}(G) \le 2$.
2. G^c *can be expressed as the union of complete graphs and of complete bipartite graphs.*
3. G *is* $(P_4, \mathrm{dart}, \ltimes)$*-free.*

This final result is quite intuitive. In contrast with Theorems 9 and 10, for a connected graph in the Hermitian case, the only forbidden subgraphs to having minimal rank ≤ 2 are easily found.

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