

## ERRATUM TO ‘A NOTE ON THE LARGEST EIGENVALUE OF NON-REGULAR GRAPHS’ \*

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**Abstract.** Let  $\lambda_1(G)$  be the largest eigenvalue of the adjacency matrix of graph  $G$  with  $n$  vertices and maximum degree  $\Delta$ . Recently,  $\Delta - \lambda_1(G) > \frac{\Delta+1}{n(3n+\Delta-8)}$  for a non-regular connected graph  $G$  was obtained in [B.L. Liu and G. Li, A note on the largest eigenvalue of non-regular graphs, *Electron J. Linear Algebra*, 17:54–61, 2008]. But unfortunately, a mistake was found in the cited preprint [T. Büyükoğlu and J. Leydold, Largest eigenvalues of degree sequences], which led to an incorrect proof of the main result of [B.L. Liu and G. Li]. This paper presents a correct proof of the main result in [B.L. Liu and G. Li], which avoids the incorrect theorem in [T. Büyükoğlu and J. Leydold].

**Key words.** Spectral radius, Non-regular graph,  $\lambda_1$ -extremal graph, Perron vector.

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**1. Introduction.** In this paper, we only consider connected, simple and undirected graphs. Let  $uv$  be an edge whose end vertices are  $u$  and  $v$ . The symbol  $N(u)$  denotes the neighbor set of vertex  $u$ . Then  $d_G(u) = |N(u)|$  is called the degree of  $u$ . The maximum degree among the vertices of  $G$  is denoted by  $\Delta$ . The sequence  $\pi = \pi(G) = (d_1, d_2, \dots, d_n)$  is called the *degree sequence* of  $G$ , where  $d_i = d_G(v)$  holds for some  $v \in V(G)$ . In the entire article, we enumerate the degrees in non-increasing order, i.e.,  $d_1 \geq d_2 \geq \dots \geq d_n$ .

Let  $A(G)$  be the adjacency matrix of  $G$ . The spectral radius of  $G$ , denoted by  $\lambda_1(G)$ , is the largest in modulus eigenvalue of  $A(G)$ . When  $G$  is connected,  $A(G)$  is irreducible and by the Perron-Frobenius Theorem (see e.g., [4]),  $\lambda_1(G)$  is a simple eigenvalue and has a unique positive unit eigenvector. We refer to such an eigenvector  $f$  as the *Perron vector* of  $G$ .

Let  $G$  be a connected non-regular graph. In [7],  $G$  is called  $\lambda_1$ -*extremal* if

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$\lambda_1(G) \geq \lambda_1(G')$  holds for any other connected non-regular graph  $G'$  with the same number of vertices and maximum degree as  $G$ . Let  $\mathcal{G}(n, \Delta)$  denote the set of all connected non-regular graphs with  $n$  vertices and maximum degree  $\Delta$ .

In [6], the following result was proved.

**THEOREM 1.1.** *Suppose  $G \in \mathcal{G}(n, \Delta)$ . Then*

$$\Delta - \lambda_1 > \frac{\Delta + 1}{n(3n + \Delta - 8)}.$$

But unfortunately, a mistake was found in the cited reference [3], resulting in an incorrect proof of Theorem 1.1. In this paper, we shall give a correct proof of Theorem 1.1, in which we avoid using the wrong theorem in reference [3].

## 2. Main result.

Let  $V_{<\Delta} = \{u : d(u) < \Delta\}$ . For the characterization of  $\lambda_1$ -extremal graphs  $G$  of  $\mathcal{G}(n, \Delta)$ , we have the following.

**THEOREM 2.1.** [6] *Suppose  $2 < \Delta < n - 1$ . If  $G$  is a  $\lambda_1$ -extremal graph of  $\mathcal{G}(n, \Delta)$ , then  $G$  must have one of the following properties:*

- (1)  $|V_{<\Delta}| \geq 2$ ,  $V_{<\Delta}$  induces a complete graph.
- (2)  $|V_{<\Delta}| = 1$ .
- (3)  $V_{<\Delta} = \{u, v\}$ ,  $uv \notin E(G)$  and  $d(u) = d(v) = \Delta - 1$ .

**DEFINITION 2.2.** [6] Suppose  $2 < \Delta < n - 1$  and  $G \in \mathcal{G}(n, \Delta)$ . Then

$G$  is called a *type-I graph* if  $G$  has property (1);

$G$  is called a *type-II graph* if  $G$  has property (2);

$G$  is called a *type-III graph* if  $G$  has property (3).

By Definition 2.2, it is easy to see the following.

**PROPOSITION 2.3.** *If  $G$  is a type-III graph, then*

$$\pi(G) = (\Delta, \Delta, \dots, \Delta, \Delta - 1, \Delta - 1),$$

where  $2 < \Delta < n - 1$ .

Suppose  $\pi = (d_1, d_2, \dots, d_n)$  and  $\pi' = (d'_1, d'_2, \dots, d'_n)$ . We write  $\pi \preceq \pi'$  if and only if  $\sum_{i=1}^n d_i = \sum_{i=1}^n d'_i$ , and  $\sum_{i=1}^j d_i \leq \sum_{i=1}^j d'_i$  for all  $j = 1, 2, \dots, n$ . Let  $C_\pi$  be the class of connected graphs with degree sequence  $\pi$ . If  $G \in C_\pi$  and  $\lambda_1(G) \geq \lambda_1(G')$  for any other  $G' \in C_\pi$ , then  $G$  is said to *have the greatest maximum eigenvalue* in  $C_\pi$ .

The next theorem (i.e., Theorem 2.3 [6]) is a crucial lemma in the proof of Theorem 1.1.

**THEOREM 2.4.** [6] *Suppose  $2 < \Delta < n - 1$  and  $G$  is a  $\lambda_1$ -extremal graph of  $\mathcal{G}(n, \Delta)$ . Then  $G$  must be a type-I or type-II graph.*

In [6], the proof of Theorem 2.4 needs the next result which was stated in [3].

**THEOREM 2.5.** [3] *Let  $\pi$  and  $\pi'$  be two distinct degree sequences with  $\pi \preceq \pi'$ . Let  $G$  be the graph with greatest maximum eigenvalue in class  $C_\pi$ , and  $G'$  in class  $C_{\pi'}$ , respectively. Then  $\lambda_1(G) < \lambda_1(G')$ .*

By Proposition 2.3, if  $G$  is a type-III graph, then  $\pi(G) = (\Delta, \Delta, \dots, \Delta, \Delta - 1, \Delta - 1)$ . Let  $G'$  be a graph with  $\pi(G') = (\Delta, \Delta, \dots, \Delta, \Delta, \Delta - 2)$ . It is easy to see that  $\pi(G) \preceq \pi(G')$ . Thus, with the application of Theorem 2.5, one can prove Theorem 2.4. But unfortunately, some counterexamples to Theorem 2.5 have been found; thus the authors of [3] have changed Theorem 2.5 from general graphs to the class of trees (see [2]).

Next we shall give a proof of Theorem 2.4 that does not depend on Theorem 2.5.

Let  $G - uv$  be the graph obtained from  $G$  by deleting the edge  $uv \in E(G)$ . Similarly,  $G + uv$  denotes the graph obtained from  $G$  by adding an edge  $uv \notin E(G)$ , where  $u, v \in V(G)$ .

**LEMMA 2.6.** (Shifting [1]) *Let  $G(V, E)$  be a connected graph with  $uv_1 \in E$  and  $uv_2 \notin E$ . Let  $G' = G + uv_2 - uv_1$ . Suppose  $f$  is the Perron vector of  $G$ . If  $f(v_2) \geq f(v_1)$ , then  $\lambda_1(G') > \lambda_1(G)$ .*

**LEMMA 2.7.** (Switching [1], [5]) *Let  $G(V, E)$  be a connected graph with  $u_1v_1 \in E$  and  $u_2v_2 \in E$ , but  $v_1v_2 \notin E$  and  $u_1u_2 \notin E$ . Let  $G' = G + v_1v_2 + u_1u_2 - u_1v_1 - u_2v_2$ . Suppose  $f$  is the Perron vector of  $G$ . If  $f(v_1) \geq f(u_2)$  and  $f(v_2) \geq f(u_1)$ , then  $\lambda_1(G') \geq \lambda_1(G)$ . The inequality is strict if and only if at least one of the two inequalities is strict.*

Let  $\mathcal{G}_1(n, \Delta)$  denote the set of all connected graphs of type-III.

**LEMMA 2.8.** *Let  $G$  be a graph in  $\mathcal{G}_1(n, \Delta)$  with  $u_1v_1 \in E(G)$ , where  $d(u_1) = \Delta$  and  $d(v_1) = \Delta - 1$ . Suppose  $f$  is the Perron vector of  $G$ . If  $f(u_1) \leq f(v_1)$ , then  $G$  cannot have the greatest maximum eigenvalue in  $\mathcal{G}_1(n, \Delta)$ .*

*Proof.* Assume that the contrary holds, i.e., suppose that  $G$  has the greatest maximum eigenvalue in  $\mathcal{G}_1(n, \Delta)$ . Without loss of generality, assume  $V_{<\Delta}(G) = \{v_1, v_2\}$ . By Definition 2.2, we have  $d(v_1) = \Delta - 1 = d(v_2)$  and  $v_1v_2 \notin E$ . We divide the proof into two cases:

Case 1.  $u_1v_2 \in E$ . Let  $G' = G + v_1v_2 - u_1v_2$ . Thus,  $d_{G'}(u_1) = \Delta - 1 = d_{G'}(v_2)$ ,  $d_{G'}(v_1) = \Delta$  and  $u_1v_2 \notin E(G')$ . Moreover,  $G'$  is also connected. Thus,  $G' \in \mathcal{G}_1(n, \Delta)$ . By Lemma 2.6, we have  $\lambda_1(G') > \lambda_1(G)$ , a contradiction.

Case 2.  $u_1v_2 \notin E$ . Since  $d_G(u_1) = \Delta > \Delta - 1 = d_G(v_1)$ , then there must exist some  $w \in N(u_1)$  such that  $w \notin N(v_1)$  and  $w \neq v_1$ . Let  $G' = G + v_1w - u_1w$ . Thus,  $d_{G'}(u_1) = d_{G'}(v_2) = \Delta - 1$ ,  $d_{G'}(v_1) = \Delta$ , and  $u_1v_2 \notin E$ . Moreover,  $G'$  is also connected. This implies that  $G' \in \mathcal{G}_1(n, \Delta)$ . By Lemma 2.6, we have  $\lambda_1(G') > \lambda_1(G)$ , a contradiction.

The result follows.  $\square$

The following is the proof of Theorem 2.4.

*Proof.* Assume that the contrary holds, i.e., suppose that there is a graph  $G$  of type-III such that  $G$  has the greatest maximum eigenvalue in  $\mathcal{G}(n, \Delta)$ . (This implies that  $G$  also has the greatest maximum eigenvalue in  $\mathcal{G}_1(n, \Delta)$ .) Without loss of generality, assume  $V_{<\Delta}(G) = \{v_1, v_2\}$ . By Definition 2.2, we have  $d(v_1) = \Delta - 1 = d(v_2)$  and  $v_1v_2 \notin E$ . Let  $f$  be the Perron vector of  $G$ . We consider the next two cases:

Case 1.  $N(v_1) = N(v_2) = \{u_1, \dots, u_{\Delta-1}\}$ . Since  $2 < \Delta < n-1$  and  $G$  is connected, there exists  $i, j$  ( $1 \leq i < j \leq \Delta - 1$ ) such that  $u_iu_j \notin E$  (otherwise, the subgraph of  $G$  induced by  $\{u_1, \dots, u_{\Delta-1}\}$  is a complete graph of order  $\Delta - 1$ , and it will yield a contradiction to the connection of  $G$  by  $\Delta < n - 1$ ).

If  $f(u_i) \leq f(v_2)$ , note that  $d(u_i) = \Delta$ ,  $d(v_2) = \Delta - 1$  and  $u_iv_2 \in E(G)$ , and by Lemma 2.8,  $G$  cannot have the greatest maximum eigenvalue in  $\mathcal{G}_1(n, \Delta)$  (also,  $G$  cannot have the greatest maximum eigenvalue in  $\mathcal{G}(n, \Delta)$ ), a contradiction. Moreover, since  $d(u_j) = \Delta$ ,  $d(v_1) = \Delta - 1$  and  $u_jv_1 \in E(G)$ , it can be proved analogously that  $f(u_j) \leq f(v_1)$  is also impossible. Thus,  $f(u_i) > f(v_2)$  and  $f(u_j) > f(v_1)$ . Let  $G' = G + u_iu_j + v_1v_2 - u_iv_1 - u_jv_2$ . Clearly,  $G'$  is also connected and  $G' \in \mathcal{G}(n, \Delta)$ . By Lemma 2.7, we can conclude that  $\lambda_1(G') > \lambda_1(G)$ , a contradiction.

Case 2.  $N(v_1) \neq N(v_2)$ . Without loss of generality, suppose  $f(v_1) \geq f(v_2)$ . Two subcases should be considered as follows.

Subcase 1.  $|N(v_1) \cap N(v_2)| \geq 1$ . Since  $N(v_1) \neq N(v_2)$ , there exists  $u_j$  such that  $u_j \in N(v_2) \setminus N(v_1)$ . Let  $G' = G + v_1u_j - v_2u_j$ . Note that  $G'$  is also connected and  $G' \in \mathcal{G}(n, \Delta)$ . By Lemma 2.6, we have  $\lambda_1(G') > \lambda_1(G)$ , a contradiction.

Subcase 2.  $|N(v_1) \cap N(v_2)| = 0$ . Since  $G$  is connected, there exists a shortest path  $P$  from  $v_2$  to  $v_1$ . Note that  $d_G(v_2) = \Delta - 1 \geq 2$  and  $|N(v_1) \cap N(v_2)| = 0$ . Then there must exist  $u_j$  such that  $u_j \in N(v_2) \setminus N(v_1)$ , but  $u_j \notin P$ . Let  $G' = G + v_1u_j - v_2u_j$ . Clearly,  $G'$  is also connected and  $G' \in \mathcal{G}(n, \Delta)$ . By Lemma 2.6, we have  $\lambda_1(G') > \lambda_1(G)$ , a contradiction.

This completes the proof.  $\square$

With the help of Theorem 2.4, it is not difficult to prove that Theorem 1.1 holds. For details of the proof, one can refer to Section 3 of [6].

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#### REFERENCES

- [1] F. Belardo, E.M.L. Marzi, and S.K. Simić. Some results on the index of unicyclic graphs. *Linear Algebra Appl.*, 416:1048–1059, 2006.
- [2] T. Buyikoğlu and J. Leydold. Graphs with given degree sequence and maximal spectral radius. *Electron. J. Combin.*, Research Paper 119, 9pp., 2008.
- [3] T. Buyikoğlu and J. Leydold. Largest eigenvalues of degree sequences. Preprint. Available at <http://arxiv.org/abs/math/0605294>. V1 May 11, 2006.
- [4] R.A. Horn and C.R. Johnson. *Matrix Analysis*. Cambridge University Press, Cambridge, 1990.
- [5] J. Leydold. A Faber-Krahn-type inequality for regular trees. *Geom. Funct. Anal.*, 7:364–378, 1997.
- [6] B.L. Liu and G. Li. A note on the largest eigenvalue of non-regular graphs. *Electron J. Linear Algebra*, 17:54–61, 2008.
- [7] B.L. Liu, J. Shen, and X.M. Wang. On the largest eigenvalue of non-regular graphs. *J. Combin. Theory Ser. B*, 97:1010–1018, 2007.