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A LOWER BOUND FOR THE SECOND LARGEST LAPLACIAN EIGENVALUE OF WEIGHTED GRAPHS*

ABRAHAM BERMAN† AND MIRIAM FARBER†

Abstract. Let G be a weighted graph on n vertices. Let $\lambda_{n-1}(G)$ be the second largest eigenvalue of the Laplacian of G. For $n \geq 3$, it is proved that $\lambda_{n-1}(G) \geq d_{n-2}(G)$, where $d_{n-2}(G)$ is the third largest degree of G. An upper bound for the second smallest eigenvalue of the signless Laplacian of G is also obtained.

Key words. Weighted graph, Laplacian matrix, Second largest eigenvalue, Lower bound, Signless Laplacian, Merris graph.

AMS subject classifications. 15A42, 05C50, 05C69.

1. Introduction. Let $G = \big(E(G), V(G)\big)$ be a simple graph (a graph without loops or multiple edges) with $|V(G)| = n$. We say that G is a *weighted graph* if it has a weight (a positive number) associated with each edge. The weight of an edge $\big\{i, j\big\} \in E(G)$ will be denoted by w_{ij}. We define the *adjacency matrix* $A(G)$ of G to be a symmetric matrix which satisfies

$$a_{ij} = \begin{cases} 0 & \text{if } \big\{i,j\big\} \notin E(G) \\ w_{ij} & \text{if } \big\{i,j\big\} \in E(G) \end{cases}.$$

The *Laplacian matrix* $L(G)$ is defined to be $D(G)-A(G)$ with $D(G) = \operatorname{diag}\big(\deg(v_1), \deg(v_2), \ldots, \deg(v_n)\big)$, where $\deg(v_i)$ is the sum of weights of all edges connected to v_i. The *signless Laplacian matrix* $Q(G)$ is defined by $D(G) + A(G)$. We denote by $0 = \lambda_1(G) \leq \lambda_2(G) \leq \cdots \leq \lambda_n(G)$ the eigenvalues of $L(G)$, and by $\mu_1(G) \leq \mu_2(G) \leq \cdots \leq \mu_n(G)$ the eigenvalues of $Q(G)$. We order the degrees of the vertices of G as $d_1(G) \leq d_2(G) \leq \cdots \leq d_n(G)$. Various bounds for the Laplacian eigenvalues of unweighted graphs, in terms of their degrees, were studied in the past (e.g., [1]). Li and Pan [6] showed that for an unweighted connected graph G with $n \geq 3$, $\lambda_{n-1}(G) \geq d_{n-1}(G)$. It is interesting to ask whether there exists a similar bound for weighted graphs. We will show it by using the following lemma ([5, p. 178]).

LEMMA 1.1. *Let A be a symmetric matrix with eigenvalues $\theta_1(G) \leq \cdots \leq \theta_n(G)$. Then* $\theta_k(A) = \max\big\{\frac{\langle Af,f\rangle}{\langle f,f\rangle} | f \perp f_{k+1}, f_{k+2}, \ldots, f_n\big\} = \min\big\{\frac{\langle Af,f\rangle}{\langle f,f\rangle} | f \perp f_1, f_2, \ldots, f_{k-1}\big\}$,

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†Mathematics Department, Technion-IIT, Haifa 32000, Israel (berman@techunix.technion.ac.il, miriamf@techunix.technion.ac.il).

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when $f_1, f_2, \ldots, f_n$ *are eigenvectors of the eigenvalues* $\theta_1, \theta_2, \ldots, \theta_n$*, respectively.*

2. The main result. We are ready now to present our main result.

THEOREM 2.1. *Let* G *be a simple weighted graph on* n *vertices with* $n \geq 3$*. Then* $\lambda_{n-1}(G) \geq d_{n-2}(G)$.

Proof. First we check the case $\lambda_{n-1}(G) = \lambda_n(G)$. Let u be the vertex with the largest degree in G. From Lemma 1.1,

$$\lambda_n(G) = \max\left\{\frac{\langle L(G)f, f\rangle}{\langle f, f\rangle}\right\}.$$

Define a vector v by

$$v_i = \begin{cases} 0 & \text{if } i \neq u \\ 1 & \text{if } i = u \end{cases}.$$

Then we have

$$\lambda_n(G) \geq \frac{\langle L(G)v, v\rangle}{\langle v, v\rangle} = d_n(G).$$

Hence, in this case, $d_{n-2}(G) \leq d_n(G) \leq \lambda_n(G) = \lambda_{n-1}(G)$. Suppose then that $\lambda_{n-1}(G) < \lambda_n(G)$. Let h be an eigenvector that corresponds to $\lambda_n(G)$. Using Lemma 1.1 we have

$$\lambda_{n-1}(G) = \max\left\{\frac{\langle L(G)f, f\rangle}{\langle f, f\rangle} | f \perp h\right\}. \tag{2.1}$$

Let s, t, q be the vertices with the largest degrees in the graph. Then there are two possibilities:

1) At least one of h_s, h_t, h_q is zero.

2) All the numbers h_s, h_t, h_q are different from zero.

In case 1), we assume without loss of generality that $h_t = 0$. Define a vector g by

$$g_i = \begin{cases} 0 & \text{if } i \neq t \\ 1 & \text{if } i = t \end{cases}.$$

Since g is orthogonal to h, we get from (2.1) that $\lambda_{n-1}(G) \geq \frac{\langle L(G)g, g\rangle}{\langle g, g\rangle}$, and hence,

$$\lambda_{n-1}(G) \geq \frac{\langle L(G)g, g\rangle}{\langle g, g\rangle} = \frac{\sum\limits_{uv \in E(G)} w_{uv}(g_u - g_v)^2}{\sum\limits_{z \in V(G)} g_z^2} = \frac{\sum\limits_{tv \in E(G)} w_{tv}(g_t - g_v)^2}{1}$$

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$$= \sum_{tv \in E(G)} w_{tv} = \deg(t) \geq \min\big\{ \deg(s), \deg(t), \deg(q) \big\} = d_{n-2}(G),$$

and we are done.

In case 2), at least two of h_s, h_t, h_q have the same sign. Suppose without loss of generality that h_s, h_t have the same sign. Define a vector g by

$$g_i = \begin{cases} 0 & \text{if } i \neq t, s \\ 1 & \text{if } i = t \\ -\delta & \text{if } i = s \end{cases}$$

with $\delta > 0$ such that g is orthogonal to h (such a positive δ exists since h_s and h_t are with the same sign). Therefore,

$$\lambda_{n-1}(G) \geq \frac{\langle L(G)g, g\rangle}{\langle g, g\rangle} = \frac{\displaystyle\sum_{uv \in E(G)} w_{uv}(g_u - g_v)^2}{\displaystyle\sum_{z \in V(G)} g_z^2}$$

$$= \frac{\displaystyle\sum_{tv \in E(G), v \neq s} w_{tv}(g_t - g_v)^2 + \sum_{us \in E(G), u \neq t} w_{us}(g_u - g_s)^2 + w_{ts}(g_t - g_s)^2}{1 + \delta^2},$$

i.e.,

$$\lambda_{n-1}(G) \geq \frac{\displaystyle\sum_{tv \in E(G), v \neq s} w_{tv} + \delta^2 \Bigg(\sum_{us \in E(G), u \neq t} w_{us} \Bigg) + w_{ts}(1 + 2\delta + \delta^2)}{1 + \delta^2}$$

$$= \frac{\deg(t) - w_{ts} + \delta^2 \big(\deg(s) - w_{ts} \big) + w_{ts}(1 + 2\delta + \delta^2)}{1 + \delta^2}$$

$$= \frac{\deg(t) + \delta^2 \deg(s) + 2w_{ts}\delta}{1 + \delta^2},$$

and since $\delta > 0$ we have:

$$\lambda_{n-1}(G) \geq \frac{\deg(t) + \deg(s)\delta^2 + 2w_{ts}\delta}{1 + \delta^2} \geq \frac{\deg(t) + \deg(s)\delta^2}{1 + \delta^2}$$

$$\geq \min\big\{ \deg(s), \deg(t) \big\} \geq \min\big\{ \deg(s), \deg(t), \deg(q) \big\} = d_{n-2}(G)$$

and we are done. ☐

REMARK 2.2. As we mentioned before, for connected unweighted graphs with $n \geq 3$, $\lambda_{n-1}(G) \geq d_{n-1}(G)$ ([6]). This is not true for weighted graphs as is shown by Figure 2.1:

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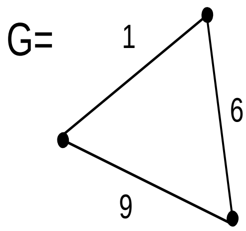

Fig. 2.1.

Note that the eigenvalues of $L(G)$ are 0, 9, 23, so $9 = \lambda_{n-1}(G) < d_{n-1}(G) = 10$.

3. Application. For a weighted graph G, we define $m_{L(G)}(I)$ to be the number of the eigenvalues of $L(G)$ that fall inside an interval I (counting multiplicities). The independence number of G is denoted by $\alpha(G)$. Merris [7] showed that if G is a simple unweighted graph on n vertices, then $m_{L(G)}([d_1(G), n]) \geq \alpha(G)$. Graphs which attain equality in the expression above were studied by Goldberg and Shapiro [4]. By similar technique to the one used by Merris in [7], we can show the following version for weighted graphs.

Theorem 3.1. *Let G be a simple weighted graph on n vertices. Then we have* $m_{L(G)}([d_1(G), \infty]) \geq \alpha(G)$.

Various examples of weighted graphs that attain equality can be found, and some of them are mentioned in [4] (for the special case of unweighted graph). This suggests the following question: Does there exist a graph for which there is no way to *assign weights* to the edges so that $m_{L(G)}([d_1(G), \infty]) = \alpha(G)$?

A first simple example is K_n $(n \geq 3)$. There is no way to assign weights to the edges of the complete graph so that $m_{L(K_n)}([d_1(K_n), \infty]) = 1$. This follows from Theorem 2.1, since

$$\lambda_n(K_n) \geq \lambda_{n-1}(K_n) \geq d_{n-2}(K_n) \geq d_1(K_n).$$

Hence, for any weighting of K_n, $m_{L(K_n)}([d_1(K_n), \infty]) \geq 2$. Are there other examples? The answer is still yes. Using Theorem 2.1, we can construct a family of such graphs in the following way: First, we take two graphs G and H, each one of them is on at least four vertices, such that $\alpha(G), \alpha(H) \leq 2$. We obtain a new graph K by adding an edge between one vertex of G and one vertex of H. If $\alpha(K) \leq 3$, then there is no way to put weights on its edges such that $m_{L(K)}([d_1(K), \infty]) = \alpha(K)$. To show it, suppose in contradiction that there is such way. We look at the graph $G \cup H$ with weights induced by K (i.e., all the edges in $G \cup H$ have the same weight as they have in K). Recall that

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$n \geq 4$, hence from Theorem 2.1 we have $\lambda_{n-1}(G) \geq d_2(G)$, $\lambda_{n-1}(H) \geq d_2(H)$, and hence $G \cup H$ has at least four eigenvalues greater than or equal to $\min\{d_2(G), d_2(H)\}$. Since $d_1(K) \leq \min\{d_2(G), d_2(H)\}$, using the interlacing theorem for adding an edge (which could be found in [3, p. 291] for unweighted graphs, but it is also true in the weighted case), we get that there are at least four eigenvalues of $L(K)$ which are above $d_1(K)$, so $\alpha(K) \geq 4$, contradicting the assumption that $\alpha(K) \leq 3$. To construct such graphs K, we can take G and H to be complete graphs (see Figure 3.1).

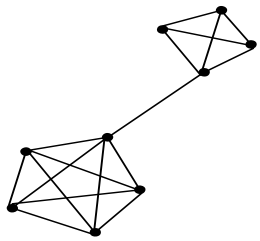

Fig. 3.1.

G and H can be chosen also to be noncomplete, but here one has to be careful in choosing the vertices. Since $\alpha(G \cup H)$=4, we must add an edge that will reduce the independence number of K to 3 (see Figure 3.2).

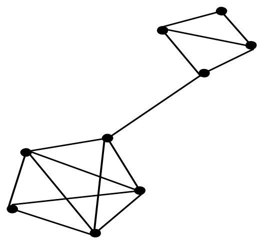

Fig. 3.2.

4. The signless Laplacian. It was proven in [2] that for a simple unweighted noncomplete graph G with n vertices ($n \geq 2$), $\mu_{n-1}(G) \geq \lambda_2(G)$. In this section, we deal with the relations between $\mu_2(G)$ and $\lambda_{n-1}(G)$. First, using techniques similar to those of the proof of Theorem 2.1, we prove the following:

Theorem 4.1. *Let G be a simple weighted graph on n vertices. Then $\mu_2(G) \leq d_3(G)$.*

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Proof. For the signless Laplacian, we have

$$\frac{\langle Q(G)g,g\rangle}{\langle g,g\rangle}=\frac{\displaystyle\sum_{uv\in E(G)} w_{uv}(g_u+g_v)^2}{\displaystyle\sum_{z\in V(G)} g_z^2}.$$

Here we denote by h an eigenvector that corresponds to $\mu_1(G)$, and hence from Lemma 1.1,

$$\mu_2(G)=\min\left\{\frac{\langle Q(G)f,f\rangle}{\langle f,f\rangle}\mid f\perp h\right\}.$$

We denote by s,t,q be the three vertices with the smallest degrees in G, and again, at least two of h_s,h_t,h_q have the same sign. We construct the vector g in the same way as in Theorem 2.1, and conclude with

$$\mu_2(G)\le\frac{\langle Q(G)g,g\rangle}{\langle g,g\rangle}$$
$$=\frac{\displaystyle\sum_{tv\in E(G),v\neq s} w_{tv}(1+0)^2+\sum_{us\in E(G),u\neq t} w_{us}(0+(-\delta))^2+w_{ts}(1+(-\delta))^2}{1+\delta^2}$$
$$=\frac{\deg(t)+\delta^2\deg(s)-2w_{ts}\delta}{1+\delta^2}\le\frac{\deg(t)+\delta^2\deg(s)}{1+\delta^2}\le d_3(G).\qquad\square$$

We conclude the paper with the following corollary, which follows directly from Theorems 2.1 and 4.1.

COROLLARY 4.2. *Let G be a simple weighted graph on n vertices, $n\ge 5$. Then $\mu_2(G)\le\lambda_{n-1}(G)$.*

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