

NEW CONDITIONS FOR THE REVERSE ORDER LAWS FOR $\{1, 3\}$ AND $\{1, 4\}$ -GENERALIZED INVERSES*

DRAGANA S. CVETKOVIĆ-ILIĆ[†]

Abstract. In this note, the reverse order laws for $\{1, 3\}$ and $\{1, 4\}$ -generalized inverses of matrices are considered. New necessary and sufficient conditions for $(AB)\{1, 3\} \subseteq B\{1, 3\} \cdot A\{1, 3\}$ and $(AB)\{1, 4\} \subseteq B\{1, 4\} \cdot A\{1, 4\}$ are presented.

Key words. Generalized inverses, $\{1, 3\}$ -Generalized inverses, $\{1, 4\}$ -Generalized inverses, MP-inverse, Reverse order law.

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1. Results. Let A be a complex matrix. We denote by $\mathcal{R}(A)$, $\mathcal{N}(A)$, $r(A)$ and $\text{nul}(A)$ the range, the null space, the rank, and the nullity of a matrix A , respectively. By P_M , we denote the orthogonal projection ($P = P^2 = P^*$) on the subspace M .

The Moore–Penrose inverse of $A \in \mathbb{C}^{n \times m}$ is the unique matrix $A^\dagger \in \mathbb{C}^{m \times n}$ satisfying the four Penrose equations in [10],

$$(1) AA^\dagger A = A, \quad (2) A^\dagger AA^\dagger = A^\dagger, \quad (3) (AA^\dagger)^* = AA^\dagger, \quad (4) (A^\dagger A)^* = A^\dagger A.$$

It is well-known that each matrix A has its Moore–Penrose inverse.

For a subset $K \subseteq \{1, 2, 3, 4\}$, we say that $B \in \mathbb{C}^{m \times n}$ is a K -inverse of $A \in \mathbb{C}^{n \times m}$ if B satisfies the Penrose equation (j) for each $j \in K$. We use AK for the collection of all K -inverses of A , and A^K for an unspecified element $X \in AK$.

The reverse order law for the Moore–Penrose inverse was first studied by Greville [5] in the 1960's, giving a necessary and sufficient condition for the reverse order law

$$(1.1) \quad (AB)^\dagger = B^\dagger A^\dagger,$$

for matrices A and B . This was followed (see [4]) by further equivalent conditions for (1.1). Sun and Wei [11] considered the reverse order law for the weighted Moore–

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[†]Department of Mathematics, University of Nis, Faculty of Science and Mathematics, 18000 Nis, Serbia (gagamaka@ptt.rs). Supported by grant no. 174007 of the Ministry of Science, Technology and Development, Republic of Serbia. Dragana Cvetković-Ilić is the recipient of UNESCO LOREAL National Scholarships Program “For Women in Science”.

Penrose inverses of two matrices. Hartwig [6] and Tian [13, 14] studied the reverse order law for the Moore–Penrose inverse of the product of three or more matrices.

The next step was to consider the reverse order law for K -inverses, where $K \subseteq \{1, 2, 3, 4\}$. The cases $K = \{1, 3\}$ and $K = \{1, 4\}$ were considered by M. Wei and Guo [16] who obtained the equivalent conditions for $B\{1, 3\}A\{1, 3\} \subseteq (AB)\{1, 3\}$, $(AB)\{1, 3\} \subseteq B\{1, 3\}A\{1, 3\}$, and $(AB)\{1, 3\} = B\{1, 3\}A\{1, 3\}$ by applying product singular value decomposition (P-SVD) of matrices. They proved that for $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times p}$, $(AB)\{1, 3\} \subseteq B\{1, 3\}A\{1, 3\}$ if and only if

$$\dim(R(Z_{14})) = \dim(R(Z_{12}, Z_{14}))$$

and

$$0 \leq \min\{p - r_2, m - r_1\} \leq n - r_1 - r_2^2 - r(Z_{14}),$$

where Z_{12}, Z_{14} and constants r_1, r_2 are described in [16, Theorem 1.1] for P-SVD of matrices A and B .

Later, also in the settings of matrices, Takane et al. [12] discovered using other techniques some new necessary and sufficient conditions for

$$B\{1, 3\}A\{1, 3\} \subseteq (AB)\{1, 3\}.$$

Djordjević [3] considered necessary and sufficient conditions for $B\{1, 3\}A\{1, 3\} \subseteq (AB)\{1, 3\}$ in the case of bounded linear operators on Hilbert spaces. Cvetković-Ilić and Harte [2] offered purely algebraic necessary and sufficient conditions for reverse order law $B\{1, 3\}A\{1, 3\} \subseteq (AB)\{1, 3\}$ for generalized inverses in C^* -algebras, extending rank conditions for matrices and range conditions for Hilbert space operators. Liu and Yang [7] derived some necessary and sufficient conditions for all three types of reverse order laws using the method of maximal and minimal rank of matrix expressions. They proved that for $A \in \mathbb{C}^{m \times n}$ and $B \in \mathbb{C}^{n \times k}$, $(AB)\{1, 3\} \subseteq B\{1, 3\}A\{1, 3\}$ if and only if

$$r(A^*AB, B) + r(A) = r(AB) + \min\{r(A^*, B), \max\{n + r(A) - m, n + r(B) - k\}\}.$$

As can be seen from the above, reverse order laws for generalized inverses have been considered in quite a number of papers. However, only [7, 16] were dealing with the particular one

$$(AB)\{1, 3\} \subseteq B\{1, 3\} \cdot A\{1, 3\}.$$

In this note, using some techniques different than those of [7, 16], we give new necessary and sufficient conditions for the inclusions:

$$(AB)\{1, 3\} \subseteq B\{1, 3\} \cdot A\{1, 3\},$$

$$(AB)\{1, 4\} \subseteq B\{1, 4\} \cdot A\{1, 4\}.$$

It is well known that the sets of $\{1, 3\}$ and $\{1, 4\}$ -generalized inverses of $A \in \mathbb{C}^{n \times m}$ are described by

$$A\{1, 3\} = \{A^\dagger + (I - A^\dagger A)Y : Y \in \mathbb{C}^{m \times n}\}$$

and

$$A\{1, 4\} = \{A^\dagger + Z(I - AA^\dagger) : Z \in \mathbb{C}^{m \times n}\}.$$

In the following, we state some auxiliary lemmas.

THEOREM 1.1. [1] *Let $A \in \mathbb{C}^{n \times m}$, $B \in \mathbb{C}^{p \times k}$, and $C \in \mathbb{C}^{n \times k}$. Then the matrix equation*

$$AXB = C$$

is consistent if and only if, for some $A^{(1)} \in A\{1\}$ and $B^{(1)} \in B\{1\}$,

$$AA^{(1)}CB^{(1)}B = C$$

in which case the general solution is

$$X = A^{(1)}CB^{(1)} + Y - A^{(1)}AYBB^{(1)}$$

for arbitrary $Y \in \mathbb{C}^{m \times p}$.

COROLLARY 1.2. *Let $B \in \mathbb{C}^{p \times k}$ and $C \in \mathbb{C}^{n \times k}$. Then the matrix equation*

$$XB = C$$

is consistent if and only if $\mathcal{N}(B) \subseteq \mathcal{N}(C)$ or, equivalently, $\mathcal{R}(C^) \subseteq \mathcal{R}(B^*)$. In that case, the general solution is given by*

$$X = CB^{(1)} + Y - YBB^{(1)}$$

for arbitrary $Y \in \mathbb{C}^{n \times p}$ and some $B^{(1)} \in B\{1\}$.

Proof. By Theorem 1.1, we see that the equation $XB = C$ is consistent if and only if $CB^{(1)}B = C$, i.e., $B^*(B^*)^{(1)}C^* = C^*$. Since $B^*(B^*)^{(1)}$ is a projection, we have that $XB = C$ is consistent if and only if $\mathcal{R}(C^*) \subseteq \mathcal{R}(B^*(B^*)^{(1)})$, i.e., $\mathcal{N}(B^{(1)}B) \subseteq \mathcal{N}(C)$. Finally, by $\mathcal{N}(B^{(1)}B) = \mathcal{N}(B)$, we get that $XB = C$ is consistent if and only if $\mathcal{N}(B) \subseteq \mathcal{N}(C)$. \square

LEMMA 1.3. *Let $A \in \mathbb{C}^{n \times m}$ and let $P \in \mathbb{C}^{m \times m}$ and $Q \in \mathbb{C}^{n \times n}$ be orthogonal projections. Then*

- (i) $(AP)^\dagger = P(AP)^\dagger$,
- (ii) $(QA)^\dagger = (QA)^\dagger Q$.

Proof. (i) It is easy to verify that $P(AP)^\dagger$ is Moore–Penrose inverse of AP . The proof for (ii) is similar. \square

LEMMA 1.4. *Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. Set $S = B^\dagger(I - A^\dagger A)$ and $C = I - A^\dagger A - S^\dagger S$. Then C is an orthogonal projection and*

$$C = (I - A^\dagger A)(I - S^\dagger S) = (I - S^\dagger S)(I - A^\dagger A) = P_{\mathcal{N}(A) \cap \mathcal{N}(B^*)}.$$

Proof. Since $SA^\dagger = 0$ and $AS^* = 0$, we get that $S^\dagger SA^\dagger A = 0$ and $A^\dagger AS^\dagger S = 0$, respectively. Hence, $C = (I - A^\dagger A)(I - S^\dagger S) = (I - S^\dagger S)(I - A^\dagger A)$. Now, it is evident that C is an orthogonal projection. We need to prove that $\mathcal{R}(C) = \mathcal{N}(A) \cap \mathcal{N}(B^*)$. Let $x \in \mathcal{N}(A) \cap \mathcal{N}(B^*)$. Then $Ax = 0$ and $B^*x = 0$ which imply that $Sx = 0$, i.e., $Cx = x$. So $x \in \mathcal{R}(C)$. On the other hand, if we suppose that $x \in \mathcal{R}(C)$, we get that $x = Cx$, i.e., $x = (I - A^\dagger A)(I - S^\dagger S)x = (I - S^\dagger S)(I - A^\dagger A)x$. Clearly, $Ax = A(I - A^\dagger A)(I - S^\dagger S)x = 0$. Also, $B^\dagger x = Sx + B^\dagger A^\dagger Ax = Sx = SCx = 0$. Hence, $B^*x = 0$, so $x \in \mathcal{N}(A) \cap \mathcal{N}(B^*)$. We conclude that $C = P_{\mathcal{N}(A) \cap \mathcal{N}(B^*)}$. \square

LEMMA 1.5. *For $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$, let $S = B^\dagger(I - A^\dagger A)$. Then the following conditions are equivalent:*

- (i) $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$,
- (ii) $(I - SS^\dagger)((AB)^\dagger A - B^\dagger A^\dagger A) = 0$,
- (iii) $(I - SS^\dagger)((AB)^\dagger A - B^\dagger) = 0$.

Proof. (i) \Rightarrow (ii) : Multiplying (i) from right by A , we get (ii).

(ii) \Rightarrow (i) : Multiplying (ii) from right by A^\dagger and using the fact that $(AB)^\dagger AA^\dagger = (AA^\dagger AB)^\dagger AA^\dagger = (AB)^\dagger$, which holds by Lemma 1.3, we get that (i) holds.

(i) \Rightarrow (iii) : Suppose that (i) holds, i.e., $(I - SS^\dagger)(AB)^\dagger = (I - SS^\dagger)B^\dagger A^\dagger$. Then

$$\begin{aligned} (I - SS^\dagger)((AB)^\dagger A - B^\dagger) &= (I - SS^\dagger)(B^\dagger A^\dagger A - B^\dagger) \\ &= (I - SS^\dagger)(-S) = 0. \end{aligned}$$

(iii) \Rightarrow (i) : Multiplying (iii) from right by A^\dagger and using the fact that $(AB)^\dagger AA^\dagger = (AB)^\dagger$, we get that (i) holds. \square

LEMMA 1.6. *Let $A \in \mathbb{C}^{n \times m}$, $B \in \mathbb{C}^{m \times k}$, $T \in \mathbb{C}^{m \times n}$, and $Z \in \mathbb{C}^{k \times n}$. Set $S = B^\dagger(I - A^\dagger A)$, $C = I - A^\dagger A - S^\dagger S$, $D = (AB)^\dagger - B^\dagger A^\dagger$ and $E = A^\dagger + S^\dagger(D + (B^\dagger B - (AB)^\dagger AB)Z) + CT$. If $(I - SS^\dagger)D = 0$, then*

$$(1.2) \quad \mathcal{N}(E) = \mathcal{N}(A^*) \cap \mathcal{N}((B^\dagger B - (AB)^\dagger AB)Z) \cap \mathcal{N}(CT).$$

Proof. Denote the right side of (1.2) by F . Suppose that $x \in \mathcal{N}(E)$. Then

$$(1.3) \quad \left(A^\dagger + S^\dagger \left(D + (B^\dagger B - (AB)^\dagger AB)Z \right) + CT \right) x = 0.$$

Multiplying (1.3) from the left side by $A^\dagger A$, we get $A^\dagger x = 0$, i.e., $x \in \mathcal{N}(A^*)$, since $AS^* = AS^\dagger = 0$ and $AC = 0$. Similarly, multiplying (1.3) by $I - A^\dagger A$ from the left side, we get

$$(1.4) \quad \left(S^\dagger \left(D + (B^\dagger B - (AB)^\dagger AB)Z \right) + CT \right) x = 0.$$

Since $(I - SS^\dagger)D = 0$, multiplying (1.4) from the left by S gives

$$(1.5) \quad \left(D + (B^\dagger B - (AB)^\dagger AB)Z \right) x = 0,$$

by Lemma 1.4 and Lemma 1.5. We already proved that $A^\dagger x = 0$. Hence, by Lemma 1.3, it follows that $Dx = 0$. Now, from (1.5), we get $(B^\dagger B - (AB)^\dagger AB)Zx = 0$. By (1.4), it follows that $CTx = 0$, so $x \in F$.

On the other hand, if $x \in F$, then $x \in \mathcal{N}(A^*)$, so $A^\dagger x = 0$ which implies that $Dx = 0$. Also, $x \in \mathcal{N}((B^\dagger B - (AB)^\dagger AB)Z) \cap \mathcal{N}(CT)$. Hence, $(B^\dagger B - (AB)^\dagger AB)Zx = CTx = 0$, which implies $x \in \mathcal{N}(E)$. \square

We now state the main result of our paper.

THEOREM 1.7. *Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. Then the following conditions are equivalent:*

- (i) $(AB)\{1, 3\} \subseteq B\{1, 3\} \cdot A\{1, 3\}$,
- (ii) $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$ and $r(C) \geq \min\{n - r(A), k - r(B)\}$,

where $S = B^\dagger(I - A^\dagger A)$ and $C = I - A^\dagger A - S^\dagger S$.

Proof. We first remark that by Lemma 1.4, we have that C is an orthogonal projection and

$$C = (I - A^\dagger A)(I - S^\dagger S) = (I - S^\dagger S)(I - A^\dagger A) = P_{\mathcal{N}(A) \cap \mathcal{N}(B^*)}.$$

(ii) \Rightarrow (i) : Suppose that (ii) holds. We must prove that for arbitrary $(AB)^{(1,3)}$ there exist $A^{(1,3)}$ and $B^{(1,3)}$ such that $(AB)^{(1,3)} = B^{(1,3)} \cdot A^{(1,3)}$. Thus, given any $Z \in \mathbb{C}^{k \times n}$, we must show that there exist $X \in \mathbb{C}^{m \times n}$ and $Y \in \mathbb{C}^{k \times m}$ such that

$$(1.6) \quad \begin{aligned} & (AB)^\dagger + (I - (AB)^\dagger(AB))Z \\ &= \left(B^\dagger + (I - B^\dagger B)Y \right) \left(A^\dagger + (I - A^\dagger A)X \right). \end{aligned}$$

Multiplying (1.6) from left first by $B^\dagger B$ and then by $(I - B^\dagger B)$ and using Lemma 1.3, we get that (1.6) implies the following:

$$(1.7) \quad (AB)^\dagger + (B^\dagger B - (AB)^\dagger(AB))Z = B^\dagger A^\dagger + B^\dagger(I - A^\dagger A)X,$$

$$(1.8) \quad (I - B^\dagger B)Z = (I - B^\dagger B)Y(A^\dagger + (I - A^\dagger A)X).$$

If we sum (1.7) and (1.8), we get (1.6). Hence, (1.6) is equivalent to (1.7) and (1.8).

Now, we have to prove that for arbitrary $Z \in \mathbb{C}^{k \times n}$, there exist $X \in \mathbb{C}^{m \times n}$ and $Y \in \mathbb{C}^{k \times m}$ such that (1.7) and (1.8) hold. Since Lemma 1.5 implies that

$$(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$$

is equivalent to

$$(I - SS^\dagger)(B^\dagger - (AB)^\dagger A) = 0,$$

we have that for each $Z \in \mathbb{C}^{k \times n}$

$$(I - SS^\dagger)\left((AB)^\dagger - B^\dagger A^\dagger + (B^\dagger B - (AB)^\dagger(AB))Z\right) = 0,$$

so equation (1.7) is solvable for each $Z \in \mathbb{C}^{k \times n}$. The set of solutions is described by

$$(1.9) \quad S_Z = \{S^\dagger\left((AB)^\dagger - B^\dagger A^\dagger + (B^\dagger B - (AB)^\dagger(AB))Z\right) + (I - S^\dagger S)T : T \in \mathbb{C}^{m \times n}\}.$$

Now, substituting X given by (1.9) in equation (1.8), we get

$$(1.10) \quad (I - B^\dagger B)Z = (I - B^\dagger B)Y\left(A^\dagger + S^\dagger\left(D + (B^\dagger B - (AB)^\dagger(AB))Z\right) + CT\right),$$

where $D = (AB)^\dagger - B^\dagger A^\dagger$. Thus, to prove (i), it is sufficient to prove that for arbitrary Z , there exist matrices P and T such that

$$(1.11) \quad (I - B^\dagger B)Z = P\left(A^\dagger + S^\dagger\left(D + (B^\dagger B - (AB)^\dagger(AB))Z\right) + CT\right),$$

which is, by Corollary 1.2, equivalent to the fact that for arbitrary Z , there exists a matrix T such that

$$(1.12) \quad \mathcal{N}(A^\dagger + S^\dagger(D + (B^\dagger B - (AB)^\dagger(AB))Z) + CT) \subseteq \mathcal{N}((I - B^\dagger B)Z).$$

Put $E = A^\dagger + S^\dagger(D + (B^\dagger B - (AB)^\dagger(AB))Z) + CT$. By Lemma 1.6, we have

$$\mathcal{N}(E) = \mathcal{N}(A^*) \cap \mathcal{N}((B^\dagger B - (AB)^\dagger(AB))Z) \cap \mathcal{N}(CT).$$

Hence, (1.12) is equivalent to

$$(1.13) \quad \mathcal{N}(A^*) \cap \mathcal{N}((B^\dagger B - (AB)^\dagger(AB))Z) \cap \mathcal{N}(CT) \subseteq \mathcal{N}((I - B^\dagger B)Z).$$

If $x \in \mathcal{N}(A^*) \cap \mathcal{N}((B^\dagger B - (AB)^\dagger AB)Z) \cap \mathcal{N}(CT)$, then $B^\dagger BZx = (AB)^\dagger ABZx$, so we conclude that for such x , the condition $x \in \mathcal{N}((I - B^\dagger B)Z)$ is equivalent to the condition $x \in \mathcal{N}((I - (AB)^\dagger AB)Z)$. Now, we get that (1.13) like (1.12) is equivalent to

$$(1.14) \quad \mathcal{N}(A^*) \cap \mathcal{N}((I - (AB)^\dagger AB)B^\dagger BZ) \cap \mathcal{N}(CT) \subseteq \mathcal{N}((I - (AB)^\dagger AB)Z).$$

Set $Q = \mathcal{N}((I - (AB)^\dagger AB)Z)$ and $Q_1 = \mathcal{N}((I - (AB)^\dagger AB)B^\dagger BZ)$. Then $Q = \{x \in \mathbb{C}^n : Zx \in \mathcal{R}(B^* A^*)\}$, $Q_1 = \{x \in \mathbb{C}^n : Zx \in \mathcal{N}(B) \oplus \mathcal{R}(B^* A^*)\}$, and $Q \subseteq Q_1$. So, to prove (i), we must show that for every matrix Z , there exists a linear mapping (matrix) T which maps the set

$$C_1 = \{x \in \mathbb{C}^n : Zx \in \mathcal{N}(B) \oplus \mathcal{R}(B^* A^*), P_{\mathcal{N}(B)} Zx \neq 0\} \cap \mathcal{N}(A^*)$$

to the set

$$(1.15) \quad C_2 = \{y \in \mathbb{C}^m : P_{\mathcal{R}(C)} y \neq 0\}.$$

In the case when $r(C) \geq n - r(A)$, there exists a linear T which maps the subspace $\mathcal{N}(A^*)$ injectively into $\mathcal{R}(C)$, thus mapping C_1 to C_2 . Now suppose that $r(C) \geq k - r(B)$.

Put $Q'_1 = Q_1 \cap \mathcal{N}(A^*)$ and $Q' = Q \cap \mathcal{N}(A^*)$. Also, denote by Z_0 the restriction of Z to the subspace Q'_1 . Let $T_0 : Q'_1 \rightarrow \mathcal{R}(C)$ be the mapping defined by $T_0 = M \circ P_{\mathcal{N}(B)} \circ Z_0$, for some injective linear mapping $M : \mathcal{N}(B) \rightarrow \mathcal{R}(C)$ and let T be any linear extension of T_0 to the space \mathbb{C}^n . Then $T(C_1) = T_0(C_1) \subseteq T_0(Q'_1) \subseteq \mathcal{R}(C)$. Let us show that $Tx \neq 0$ on $C_1 = Q'_1 \setminus Q'$. If $Tx = 0$ for some $x \in Q'_1 \setminus Q'$, it follows that $T_0(x) = 0$, so $(P_{\mathcal{N}(B)} \circ Z_0)(x) = 0$ which implies that $x \in Q'$ which is a contradiction. Hence, $T(C_1) \subseteq \mathcal{R}(C)$ and $T(x) \neq 0$ for every $x \in C_1$ and hence, $T(C_1) \subseteq C_2$.

(i) \Rightarrow (ii) : If (i) holds, then for arbitrary $Z \in \mathbb{C}^{k \times n}$, there exist $X \in \mathbb{C}^{m \times n}$ and $Y \in \mathbb{C}^{k \times m}$ such that (1.6) holds.

Multiplying (1.6) by $B^\dagger B$ from the left, we get that

$$(AB)^\dagger + (B^\dagger B - (AB)^\dagger AB)Z = B^\dagger A^\dagger + B^\dagger(I - A^\dagger A)X.$$

For $Z = 0$, we get that the equation

$$(AB)^\dagger - B^\dagger A^\dagger = B^\dagger(I - A^\dagger A)X$$

is solvable. Hence,

$$\mathcal{R}((AB)^\dagger - B^\dagger A^\dagger) \subseteq \mathcal{R}(S)$$

which is equivalent to

$$(1.16) \quad (I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0.$$

Using the previous part of the proof, we get that if (1.16) and (i) hold, then for arbitrary Z , there exists a matrix T which maps the set $C_1 = \{x \in \mathbb{C}^n : Zx \in \mathcal{N}(B) \oplus \mathcal{R}(B^*A^*), P_{\mathcal{N}(B)}Zx \neq 0\} \cap \mathcal{N}(A^*)$ to the set $C_2 = \{y \in \mathbb{C}^m : P_{\mathcal{R}(C)}y \neq 0\}$.

We distinguish two cases:

1° If $\dim(\mathcal{N}(A^*)) \leq \dim(\mathcal{N}(B))$, then there exists Z such that $C_1 = \mathcal{N}(A^*) \setminus \{0\}$. It is now easy to see that there exists T which maps C_1 into the set C_2 defined by (1.15) if and only if $\dim \mathcal{N}(A^*) \leq \dim(\mathcal{R}(C))$.

2° Suppose now $\dim(\mathcal{N}(A^*)) > \dim(\mathcal{N}(B))$. Pick a subspace K of $\mathcal{N}(A^*)$ of dimension $\text{nul}(B)$, a linear $Z : \mathbb{C}^k \rightarrow \mathbb{C}^n$ mapping K isomorphically to $\mathcal{N}(B)$ and let $T : \mathbb{C}^k \rightarrow \mathbb{C}^m$ be as supposed to exist. Clearly, we have $K \setminus \{0\} \subseteq C_1$. Thus, $(P_{\mathcal{R}(C)} \circ T)x \neq 0$ for $x \in K \setminus \{0\}$, i.e., $P_{\mathcal{R}(C)} \circ T$ maps injectively K into $\mathcal{R}(C)$. This proves $\dim(\mathcal{N}(B)) = \dim K \leq \dim(\mathcal{R}(C))$. \square

REMARK 1.8.

(1) The first condition from Theorem 1.7 (ii):

$$(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$$

has a few equivalent forms which are given in Lemma 1.5. Beside these forms it is equivalent to:

$$(I - SS^\dagger)((AB)^\dagger AB - B^\dagger B) = 0.$$

(2) The second condition from Theorem 1.7 (ii):

$$r(C) \geq \min\{n - r(A), k - r(B)\},$$

can be written as

$$r(C) \geq \min\{\text{nul}(A^*), \text{nul}(B)\}.$$

(3) Since $C = P_{\mathcal{N}(A) \cap \mathcal{N}(B^*)}$, we have that $r(C) = \dim(\mathcal{N}(A) \cap \mathcal{N}(B^*))$.

The proof of the part (i) \Rightarrow (ii) of Theorem 1.7 yields the following:

COROLLARY 1.9. *Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. Then the following conditions are equivalent:*

$$(i^*) \quad (AB)\{1, 3\} \subseteq B\{1, 3\} \cdot A\{1, 3\},$$

$(ii^*) (I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$, and at least one of the two conditions below holds:

- (a) $r(C) \geq k - r(B)$, $k - r(B) < n - r(A)$,
- (b) $r(C) \geq n - r(A)$, $k - r(B) \geq n - r(A)$,

where $S = B^\dagger(I - A^\dagger A)$ and $C = I - A^\dagger A - S^\dagger S$.

Proof. $(ii^*) \Rightarrow (i^*)$: If any of the conditions (a) and (b) holds, then we have that $r(C) \geq \min\{n - r(A), k - r(B)\}$, so the condition (ii) from Theorem 1.7 is satisfied, which implies (i^*) .

$(i^*) \Rightarrow (ii^*)$: If (i^*) holds, then, by Theorem 1.7, we have that $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$. Using the part $(ii) \Rightarrow (i)$ of the proof of Theorem 1.7, we get that for arbitrary Z , there exists a matrix T which maps the set $C_1 = \{x \in \mathbb{C}^n : Zx \in \mathcal{N}(B) \oplus \mathcal{R}(B^* A^*), P_{\mathcal{N}(B)} Zx \neq 0\} \cap \mathcal{N}(A^*)$ into the set $C_2 = \{y \in \mathbb{C}^m : P_{\mathcal{R}(C)} y \neq 0\}$.

Now, as in the proof of that theorem we distinguish two cases:

a) Suppose that $\dim(\mathcal{N}(A^*)) > \dim(\mathcal{N}(B))$, i.e., $k - r(B) < n - r(A)$. Pick a subspace K of $\mathcal{N}(A^*)$ of dimension $\text{nul}(B)$, a linear $Z : \mathbb{C}^k \rightarrow \mathbb{C}^n$ mapping K isomorphically to $\mathcal{N}(B)$ and let $T : \mathbb{C}^k \rightarrow \mathbb{C}^m$ be as supposed to exist. Clearly, we have $K \setminus \{0\} \subseteq C_1$. Thus, $(P_{\mathcal{R}(C)} \circ T)x \neq 0$ for $x \in K \setminus \{0\}$, i.e., $P_{\mathcal{R}(C)} \circ T$ maps injectively K into $\mathcal{R}(C)$. This proves $\dim(\mathcal{N}(B)) = \dim K \leq \dim(\mathcal{R}(C))$, i.e., $r(C) \geq k - r(B)$. \square

b) If $\dim(\mathcal{N}(A^*)) \leq \dim(\mathcal{N}(B))$, i.e., $k - r(B) \geq n - r(A)$, then there exists Z such that $C_1 = \mathcal{N}(A^*) \setminus \{0\}$. Hence, the existence of mapping T which maps C_1 to the set C_2 is equivalent to $\dim \mathcal{N}(A^*) \leq \dim(\mathcal{R}(C))$, i.e., $r(C) \geq n - r(A)$.

Now, we conclude that (ii^*) holds. \square

COROLLARY 1.10. Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. If $m < n$ and $m < k$, then

$$(AB)\{1, 3\} \subseteq B\{1, 3\} \cdot A\{1, 3\},$$

cannot be satisfied.

Proof. Since $r(C) = \dim(\mathcal{N}(A) \cap \mathcal{N}(B^*))$, we have that $r(C) \leq m - r(A)$ and $r(C) \leq m - r(B)$. If $m < n$ and $m < k$, then neither of the conditions (a) and (b) from Corollary 1.9 can be satisfied. \square

The case $K = \{1, 4\}$ is treated completely analogously and the corresponding result follows by taking adjoints, or by reversal of products:

THEOREM 1.11. Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. Then the following conditions are equivalent:

- (i') $(AB)\{1, 4\} \subseteq B\{1, 4\} \cdot A\{1, 4\}$,
 (ii') $((AB)^\dagger - B^\dagger A^\dagger)(I - V^\dagger V) = 0$ and $r(D) \geq \min\{n - r(A), k - r(B)\}$,

where $V = (I - BB^\dagger)A^\dagger$ and $D = I - BB^\dagger - VV^\dagger$.

COROLLARY 1.12. Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. Then the following conditions are equivalent:

- (i'') $(AB)\{1, 4\} \subseteq B\{1, 4\} \cdot A\{1, 4\}$,
 (ii'') $((AB)^\dagger - B^\dagger A^\dagger)(I - V^\dagger V) = 0$ and at least one of the two conditions below holds:

- (a') $r(D) \geq n - r(A)$, $n - r(A) < k - r(B)$
 (b') $r(D) \geq k - r(B)$, $n - r(A) \geq k - r(B)$,

where $V = (I - BB^\dagger)A^\dagger$ and $D = I - BB^\dagger - VV^\dagger$.

COROLLARY 1.13. Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. If $m < n$ and $m < k$, then

$$(AB)\{1, 4\} \subseteq B\{1, 4\} \cdot A\{1, 4\}$$

cannot be satisfied.

It is interesting that in Theorem 1.7 and Theorem 1.11, the matrices C and D are equal. Furthermore, $C = D = P_{\mathcal{N}(A) \cap \mathcal{N}(B^*)}$ and $r(C) = r(D) = \dim(\mathcal{N}(A) \cap \mathcal{N}(B^*))$. Also, the second condition from (ii) of Theorem 1.7 and the second condition of (ii') of Theorem 1.11 are exactly the same. So, we have the following result.

THEOREM 1.14. Let $A \in \mathbb{C}^{n \times m}$ and $B \in \mathbb{C}^{m \times k}$. Then the following conditions are equivalent:

- (i) $(AB)\{1, 3\} \subseteq B\{1, 3\} \cdot A\{1, 3\}$, $(AB)\{1, 4\} \subseteq B\{1, 4\} \cdot A\{1, 4\}$,
 (ii) $(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) = 0$, $((AB)^\dagger - B^\dagger A^\dagger)(I - V^\dagger V) = 0$, and
 $r(C) \geq \min\{n - r(A), k - r(B)\}$,

where $S = B^\dagger(I - A^\dagger A)$, $V = (I - BB^\dagger)A^\dagger$ and $C = I - A^\dagger A - S^\dagger S$.

2. Numerical examples. The following examples illustrate application of the presented results.

EXAMPLE 2.1. Let

$$A = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 111 & 0 \\ 10 & -10 & -4 & -4 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & -1 & 1 \\ -1 & -1 & -1 \end{bmatrix}.$$

We have that $r(A) = 3$, $r(B) = 2$, and $r(C) = 0$. Using the program **Mathematica**, we can easily check that

$$(I - SS^\dagger)((AB)^\dagger - B^\dagger A^\dagger) \neq 0,$$

while

$$((AB)^\dagger - B^\dagger A^\dagger)(I - V^\dagger V) = 0, \quad r(C) \geq n - r(A) \quad \text{and} \quad k - r(B) \geq n - r(A).$$

Hence, $(AB)\{1, 3\} \subseteq B\{1, 3\} \cdot A\{1, 3\}$ is not satisfied while $(AB)\{1, 4\} \subseteq B\{1, 4\} \cdot A\{1, 4\}$.

EXAMPLE 2.2. Let

$$A = \begin{bmatrix} 5 & 5 & 5 & 5 & 5 & 6 \\ 5 & 1 & 111 & 0 & 2 & 3 \\ 1111 & 23450 & -4 & -4 & 3 & 2 \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} 1 & 1543 & 1 \\ 1 & 1234 & 213 \\ 1 & -1 & 1 \\ -1 & -1 & -1 \\ 1 & 2 & 3 \\ 1 & 2 & 3 \end{bmatrix}.$$

We have that $r(A) = r(B) = 3$ and $r(C) = 0$. Furthermore, all the conditions in Theorem 1.14 are satisfied and hence,

$$(AB)\{1, 3\} \subseteq B\{1, 3\} \cdot A\{1, 3\} \quad \text{and} \quad (AB)\{1, 4\} \subseteq B\{1, 4\} \cdot A\{1, 4\}.$$

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