SOME RESULTS ON THE LARGEST AND LEAST 
EIGENVALUES OF GRAPHS∗

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1. Introduction. Throughout this paper, we consider only simple graphs, herein called just graphs. Let $G = (V, E)$ be a graph on vertex set $V = \{v_1, v_2, \ldots, v_n\}$ and edge set $E = E(G)$. The distance between vertices $u$ and $v$ is denoted by $d(u, v)$. The diameter of a graph is the maximal distance between any two vertices. The adjacency matrix of a graph $G$ is denoted by $A(G)$ and defined as the $n \times n$ matrix $(a_{ij})$, where $a_{ij} = 1$ if $v_iv_j \in E(G)$ and 0 otherwise. Since $A(G)$ is a real symmetric matrix, its eigenvalues must be real, and may be ordered as $\lambda_1(G) \geq \lambda_2(G) \geq \cdots \geq \lambda_n(G)$.

The largest eigenvalue $\lambda_1(G)$ or $\rho(G)$ is called the spectral radius or the index of $G$. By Perron-Frobenius Theorem, $\lambda_1(G)$ is simple and has a unique positive unit eigenvector corresponding to it. We will refer to such an eigenvector as the Perron vector of $G$. It is known that $\lambda_n(G) = -\lambda_1(G)$ for a bipartite graph $G$ (see [7]). A unit eigenvector corresponding to $\lambda_n(G)$ is called a least vector of $G$. 

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A kite $K_{i,n-\omega}$ is the graph obtained from a complete graph $K_{\omega}$ and a path $P_{n-\omega}$ by adding an edge between a vertex of $K_{\omega}$ and an end point of $P_{n-\omega}$. A matching in a graph is a set of disjoint edges, and the maximum cardinality of a matching over all possible matching in a graph $G$ is called the matching number of $G$, denoted by $\mu$.

In a susceptible-infectious-susceptible (SIS) type of network infection, the steady-state infection of the network is determined by a phase transition at the epidemic threshold $\tau_c = \frac{1}{\lambda_1(A)}$: When the effective infection rate $\tau > \tau_c$, the network is infected, whereas below $\tau_c$, the network is virus free. Motivated by a threshold separating two different phases of a dynamic process on a network, we want to change the network in order to enlarge the network’s epidemic threshold $\tau_c$, or, equivalently, to lower $\lambda_1(A)$. We are searching for a strategy so that, after removing $k$ vertices, $\lambda_1(A)$ is minimal. Recently, Li et al. [13] presented a lower bound for the spectral radius of a graph in which some vertices are removed, and Mieghem et al. [16] gave lower and upper bounds for the spectral radius of a graph when some edges are removed. Naturally, Xing and Zhou [23] established an upper bound for the least eigenvalue of a graph when some vertices are removed using the components of the least vector. Furthermore, the authors [23] also gave lower and upper bounds for the least eigenvalue of a graph when some edges are removed. In Section 2 of this paper, we consider the case of connected graphs, and present an incomparable sharp lower bound for the spectral radius of a graph and an incomparable sharp upper bound for the least eigenvalue of a graph when some vertices are removed.

In [1, 2], Aouchiche et al. gave the following conjectures involving index, diameter and matching number of $G$ (see also [3]).

**Conjecture 1.1** ([4]). Let $G$ be a connected graph with diameter $D$. Then

$$\lambda_1(G) + D \leq n - 1 + 2 \cos \frac{\pi}{n+1},$$

and equality holds if and only if $G \cong P_n$.

**Conjecture 1.2** ([1, 2, 3]). Let $G$ be a connected graph with matching number $\mu$. Then

$$\lambda_1(G) - \mu \leq n - 1 - \lfloor n/2 \rfloor,$$

and equality holds if and only if $G \cong K_n$.

**Conjecture 1.3** ([1, 2, 3]). Let $G$ be a connected graph with matching number $\mu$. Then

$$\frac{\lambda_1(G)}{\mu} \leq \sqrt{n-1},$$

and equality holds if and only if $G \cong K_{1,n-1}$. 
D. Stevanović [20] proved Conjectures 1.2 and 1.3. However, we observed that the extremal graphs in the statement of Conjecture 1.3 are not complete. Moreover, Stevanovic’s theorem is also missing some extremal graphs. When \( n = 5 \), \( K_5 \) is also the extremal graph in Conjecture 1.3 which is not considered in [20]. In Section 3 of this paper, we show that Conjecture 1.1 is right, and Conjectures 1.2 and 1.3 still hold when removing the condition that \( G \) is connected.

Recently, researchers have paid much attention to the least eigenvalues of graphs with a given value of graph invariant, for instance: order and size [5, 6, 10, 19], unicyclic graphs with a given number of pendant vertices [15], matching number and independence number [21], number of cut vertices [22], connectivity [24], domination number [25]. A connected graph \( G \) is called a quasi-tree graph if there exists \( v \in V(G) \) such that \( G - v \) is a tree. H. Liu and M. Lu [14] determined the maximal and the second maximal spectral radii among all quasi-tree graphs. In section 4, we characterize the extremal graph which attains the minimum least eigenvalue among all quasi-tree graphs.

2. On the largest and least eigenvalues of graphs when vertices are removed.

**Theorem 2.1.** Let \( G \) be a connected graph with \( V(G) = V_1 \cup V_2 = \{v_1, \ldots, v_n\} \) where \( V_1 = \{v_1, \ldots, v_k\} \) and \( V_2 = \{v_{k+1}, \ldots, v_n\} \), and \( G[V_i] \) be the induced subgraphs of \( G \) for \( i = 1, 2 \). Suppose that \( A \) and \( A_i \) are the adjacency matrices of \( G \) and \( G[V_i] \) for \( i = 1, 2 \), respectively. Let \( X = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)^t \) be the Perron vector of \( G \), where \( x_i \) corresponds to \( v_i \) for \( i = 1, \ldots, n \). Then

\[
\lambda_1(A_1) \geq \lambda_1(A) - \frac{\sum_{v_i, v_j \in E[V_1, V_2]} x_i x_j}{\sum_{v_i \in V_1} x_i^2},
\]

and equality holds if and only if \( X_1 = (x_1, \ldots, x_k)^t \) is an eigenvector of \( G_1 \) corresponding to \( \lambda_1(A_1) \).

**Proof.** Let \( A = \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} \) and \( X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \) where \( X_1 = (x_1, \ldots, x_k)^t \) and \( X_2 = (x_{k+1}, \ldots, x_n)^t \). Since \( AX = \lambda_1(A)X \), thus

\[
\begin{cases}
\lambda_1(A)X_1 = A_1X_1 + B_1X_2, \\
\lambda_1(A)X_2 = B_2X_1 + A_2X_2.
\end{cases}
\]  

(2.1)

Note that

\[
X_1^t B_1 X_2 = \sum_{i=1}^k \sum_{j=k+1}^n x_i a_{ij} x_j = \sum_{v_i, v_j \in E[V_1, V_2]} x_i x_j
\]
and by the second equation in (2.1),
\[ X_1^t B_2 X_1 + X_2^t A_2 X_2 = \lambda_1(A) X_2^t X_2. \]

Thus,
\[
\lambda_1(A) = X^t A X = (X_1^t, X_2^t) \begin{pmatrix} A_1 & B_1 \\ B_2 & A_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix}
= X_1^t A_1 X_1 + X_2^t B_1 X_1 + X_1^t B_1 X_2 + X_2^t A_2 X_2
= X_1^t A_1 X_1 + \sum_{v_i, v_j \in E(V_1, V_2)} x_i x_j + \lambda_1(A) X_2^t X_2.
\]

Note that \( X_1^t X_1 + X_2^t X_2 = 1 \), then
\[
\lambda_1(A) \geq \frac{X_1^t A_1 X_1}{X_1^t X_1} = \frac{\lambda_1(A) - \lambda_1(A) X_2^t X_2 - \sum_{v_i, v_j \in E(V_1, V_2)} x_i x_j}{X_1^t X_1}
= \lambda_1(A) - \frac{\sum_{v_i, v_j \in E(V_1, V_2)} x_i x_j}{\sum_{v_i \in V_1} x_i^2}.
\]

Equality holds if and only if \( X_1 \) is an eigenvector of \( G_1 \) corresponding to \( \lambda_1(A_1) \).

For any graph \( G \) (not necessarily connected) with \( V(G) = V_1 \cup V_2 \) and corresponding graphs \( G[V_i] \) for \( i = 1, 2 \) are the induced subgraphs of \( G \). Suppose that \( A \) and \( A_i \) are the adjacency matrices of \( G \) and \( G[V_i] \) for \( i = 1, 2 \), respectively. C. Li et al. [13] gave a lower bound on \( \lambda_1(A_1) \), that is,
\[
\lambda_1(A_1) \geq \left(1 - \frac{2}{n} \sum_{v_i \in V_2} x_i^2\right) \lambda_1(A) + \sum_{v_i, v_j \in E(G[V_1])} x_i x_j,
\]
where \( X = (x_1, \ldots, x_n)^t \) is the eigenvector of \( G \) corresponding to \( \lambda_1(G) \). Let \( G_i = G - v_i \). Then \( \lambda_1(G_1) \geq (1 - 2x_i^2) \lambda_1(G) \). Nikiforov [17] improved the lower bound of \( \lambda_1(G_1) \). When \( G \) is connected, we provide a necessary and sufficient condition for the lower bound is attained (see Theorem 2.2).

**Theorem 2.2** ([17]). Let \( G \) be a connected graph with order \( n \). Let \( v_i \in V(G) \) and \( G_i = G - v_i \). Suppose that \( X = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n, x_i)^t \) is the Perron vector of \( G \) corresponding to \( \lambda_1(G) \). Then
\[
\lambda_1(G_i) \geq \lambda_1(G) \frac{1 - 2x_i^2}{1 - x_i^2}
\]
and equality holds if and only if \( X_i = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)^t \) is an eigenvector of \( G_i \) corresponding to \( \lambda_1(G_i) \).
Obviously, the equality holds if and only if \( X \) and equality holds if and only if \( G \). Note that \( \lambda_i(G) \) is an eigenvector of \( G \) corresponding to \( G \). Since \( AX = \lambda_i(A)X \), thus we have \( \lambda_i(A)x_i = b^tX_1 \). Therefore,

\[
\lambda_i(A) = X^tAX = (X^t_1 x_i) \begin{pmatrix} A_1 & b \\ b^t & 0 \end{pmatrix} \begin{pmatrix} X_1 \\ x_i \end{pmatrix} = X^t_1A_1X_1 + x_ib^tX_1 + X^t_1bx_i = X^t_1A_1X_1 + 2\lambda_i(A)x_i^2.
\]

Note that \( X^t_1X_1 + x_i^2 = 1 \), then

\[
\lambda_i(A_1) \geq \frac{X^t_1A_1X_1}{X^t_1X_1} = \lambda_i(A) \frac{1 - 2x_i^2}{1 - x_i^2}.
\]

Obviously, the equality holds if and only if \( X_1 = (x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)^t \) is an eigenvector of \( G \) corresponding to \( \lambda_i(G) \).

**Corollary 2.3.** Let \( G \) be a connected graph with order \( n \). Suppose that \( X = (x_1, x_2, \ldots, x_n)^t \) is the Perron vector of \( G \), where \( x_1 \geq x_2 \geq \cdots \geq x_n \). Then \( \max_{i=1}^n \lambda_i(G_i) \geq \lambda_i(G)^{n-2}n^{-1} \), and the equality holds if and only if \( G \cong K_n \). Meanwhile, \( \min_{i=1}^n \lambda_i(G_i) \geq 0 \), and the equality holds if and only if \( G \cong K_{1,n-1} \).

**Proof.** Let \( X = (x_1, x_2, \ldots, x_n)^t \) be the Perron vector of \( G \) where \( x_i \) corresponds to \( v_i \) for \( i = 1, \ldots, n \). It is easy to see that \( f(x) = \frac{1 - 2x^2}{1 - x^2} \) is a decreasing function when \( 0 < x < 1 \). So \( \lambda_i(G_i) \) attains the maximum if \( v_i = v_n \) and the minimum if \( v_i = v_1 \). Note that \( \frac{1}{\sqrt{n}} \leq x_1 \leq \frac{1}{\sqrt{2}} \) and \( 0 < x_n \leq \frac{1}{\sqrt{n}} \). Therefore \( \lambda_i(G_1) \geq 0 \) and the equality holds if and only if \( x_1 = \frac{1}{\sqrt{2}} \), that is \( G \cong K_{1,n-1} \). On the other hand, \( \lambda_i(G_n) \geq \lambda_i(G)^{n-2}n^{-1} \) and the equality holds if and only if \( x_n = \frac{1}{\sqrt{n}} \). Then \( x_i = \frac{1}{\sqrt{n}} \) for \( i = 1, \ldots, n \) and then \( G \) is a regular graph. By Theorem 2.2, the equality holds if and only if \( X_1 = (x_1, \ldots, x_{n-1})^t = \left( \frac{1}{\sqrt{n}}, \ldots, \frac{1}{\sqrt{n}} \right)^t \) is the eigenvector of \( G_n \) corresponding to \( \lambda_i(G_n) \), that is \( G_n \) is also a regular graph. Therefore, it is easy to see that \( G \cong K_n \).

**Theorem 2.4.** Let \( G \) be a connected graph with \( V(G) = V_1 \cup V_2 = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\} \) where \( V_1 = \{v_1, \ldots, v_k\} \) and \( V_2 = \{v_{k+1}, \ldots, v_n\} \) and \( G_i = G[V_i] \) be the subgraphs of \( G \) for \( i = 1, 2 \). Suppose that \( A \) and \( A_i \) are the adjacency matrices of \( G \) and \( G_i \) for \( i = 1, 2 \), respectively. Let \( X = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)^t \) be a least vector of \( G \) where \( x_i \) corresponds to \( v_i \) for \( i = 1, \ldots, n \). Then

\[
\lambda_n(A_1) \leq \frac{\lambda_n(A) \left( 1 - 2 \sum_{v \in V_2} x_v^2 \right) + \sum_{v \in V_2} \sum_{v' \in E(G_2)} x_v x_{v'}}{1 - \sum_{v \in V_2} x_v^2},
\]

and equality holds if and only if \( X_1 = (x_1, \ldots, x_k)^t \) is a least vector of \( G_1 \).
Some Results on the Largest and Least Eigenvalues of Graphs

Let \( A = \begin{pmatrix} A_1 & B \\ B^t & A_2 \end{pmatrix} \) and \( X = \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} \) where \( X_1 = (x_1, \ldots, x_k)^t \) and
\( X_2 = (x_{k+1}, \ldots, x_n)^t \). Since \( AX = \lambda_n(A)X \), thus
\[
\begin{align*}
\lambda_n(A)X_1 &= A_1X_1 + BX_2, \\
\lambda_n(A)X_2 &= B^tX_1 + A_2X_2.
\end{align*}
\]
Then
\[
\lambda_n(A) = X^tAX = (X_1^t X_2^t) \begin{pmatrix} A_1 & B \\ B^t & A_2 \end{pmatrix} \begin{pmatrix} X_1 \\ X_2 \end{pmatrix} = X_1^t A_1 X_1 + X_2^t B^t X_1 + X_1^t B X_2 + X_2^t A_2 X_2 = X_1^t A_1 X_1 + 2X_2^t (B^t X_1 + A_2 X_2) - X_2^t A_2 X_2 = X_1^t A_1 X_1 + 2\lambda_n(A)X_2^t X_2 - X_2^t A_2 X_2.
\]
Note that \( X_1^t X_1 + X_2^t X_2 = 1 \), then
\[
\lambda_n(A_1) \leq \frac{X_1^t A_1 X_1}{X_1^t X_1} = \frac{\lambda_n(A) - 2\lambda_n(A)X_2^t X_2 + \sum_{v_i, v_j \in E(G_2)} x_i x_j}{X_1^t X_1} = \frac{\lambda_n(A) (1 - 2 \sum_{v_i \in V_2} x_i^2) + \sum_{v_i, v_j \in E(G_2)} x_i x_j}{1 - \sum_{v_i \in V_2} x_i^2}.
\]
Equality holds if and only \( X_1 \) is a least vector of \( A_1 \) corresponding to \( \lambda_n(A_1) \).

**Corollary 2.5.** Let \( G \) be a connected graph with \( V(G) = V_1 \cup V_2 = \{v_1, \ldots, v_k, v_{k+1}, \ldots, v_n\} \) where \( V_1 = \{v_1, \ldots, v_k\} \) and \( V_2 = \{v_{k+1}, \ldots, v_n\} \) and \( G_i = G[V_i] \) be the subgraphs of \( G \) for \( i = 1, 2 \). Suppose that \( A \) and \( A_i \) are the adjacency matrices of \( G \) and \( G_i \) for \( i = 1, 2 \), respectively. Let \( X = (x_1, \ldots, x_k, x_{k+1}, \ldots, x_n)^t \) be a least vector of \( G \) where \( x_i \) corresponds to \( v_i \) for \( i = 1, \ldots, n \). If \( x_i = 0 \) for \( i = k+1, \ldots, n \), then \( \lambda_n(A) = \lambda_n(A_1) \).

**Proof.** By Theorem 2.4, \( \lambda_n(A_1) \leq \lambda_n(A) \). On the other hand, by Cauchy interlacing theorem, \( \lambda_n(A_1) \geq \lambda_n(A) \). Thus, \( \lambda_n(A_1) = \lambda_n(A) \).

**3. On conjectures involving the spectral radius of graphs.** The following inequalities are well-known Courant-Weyl inequalities.

**Lemma 3.1.** Let \( A \) and \( B \) be \( n \times n \) Hermitian matrices and \( C = A + B \). Then
\[
\lambda_i(C) \leq \lambda_j(A) + \lambda_{i-j+1}(B) \quad (n \geq i \geq j \geq 1),
\]
\[
\lambda_i(C) \geq \lambda_j(A) + \lambda_{i-j+n}(B) \quad (1 \leq i \leq j \leq n).
\]
Similar to the Courant-Weyl inequalities, we have the following result.

**Lemma 3.2.** Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Suppose $G_1$ and $G_2$ are two subgraphs of $G$ such that $V(G) = V(G_1) \cup V(G_2)$ and $E(G) = E(G_1) \cup E(G_2)$ with $E(G_1), E(G_2) \neq \emptyset$. Then $\lambda_1(G) \leq \lambda_1(G_1) + \lambda_1(G_2)$, equality holds if and only if $G, G_1$ and $G_2$ have a common eigenvector corresponding to $\lambda_1(G)$, $\lambda_1(G_1)$ and $\lambda_1(G_2)$.

**Proof.** Let $A$, $A_1$ and $A_2$ be the adjacent matrices of $G$, $G_1$ and $G_2$, respectively. Obviously, $A = A_1 + A_2$. Suppose that $X$ is an eigenvector of $A$ corresponding to $\lambda_1(A)$, then

$$\lambda_1(A) = X^TAX = X^TA_1X + X^TA_2X \leq \lambda_1(A_1) + \lambda_1(A_2).$$

If $\lambda_1(A) = \lambda_1(A_1) + \lambda_1(A_2)$, then $\lambda_1(A_1) = X^TA_1X$ and $\lambda_1(A_2) = X^TA_2X$, that is, $X$ is a common eigenvector of $A_1$ and $A_2$ corresponding to $\lambda_1(A_1)$ and $\lambda_1(A_2)$. For the converse, suppose that $X$ is a common eigenvector of $A$, $A_1$ and $A_2$ corresponding to $\lambda_1(G)$, $\lambda_1(G_1)$ and $\lambda_1(G_2)$, then it is easy to see that $\lambda_1(A_1) = X^TA_1X$ and $\lambda_1(A_2) = X^TA_2X$. Therefore $\lambda_1(A) = \lambda_1(A_1) + \lambda_1(A_2)$. \hfill \Box

By the above lemma, we get the following corollary.

**Corollary 3.3.** Let $G$ be a graph with vertex set $V(G)$ and edge set $E(G)$. Suppose $G_1$ and $G_2$ are two subgraphs of $G$ such that $V(G_1) = V(G_2) = V(G)$ and $E(G) = E(G_1) \cup E(G_2)$ with $E(G_1), E(G_2) \neq \emptyset$. If one of $G_1$ and $G_2$ has an isolated vertex, and the other is connected, then $\lambda_1(G) < \lambda_1(G_1) + \lambda_1(G_2)$.

**Proof.** Without loss of generality, we may assume that $G_1$ is connected and $G_2$ contains an isolated vertex, say $u$. Since $G_1$ is connected, by Perron-Frobenius Theorem, the eigenvector, say $X = (x_1, \ldots, x_n)$ of $G_1$ corresponding to $\lambda_1(G_1)$ is positive, that is $x_i > 0$ for $i = 1, \ldots, n$. By Lemma 3.2, $\lambda_1(G) \leq \lambda_1(G_1) + \lambda_1(G_2)$. If $\lambda_1(G) = \lambda_1(G_1) + \lambda_1(G_2)$, then $X$ is also the eigenvector of $G_2$ corresponding to $\lambda_1(G_2)$. Then $A(G_2)X = \lambda_1(G_2)X$. Since $u$ is an isolated vertex of $V(G_2)$, we have $\lambda_1(G_2)X_u = \lambda_1(G_2)x_u > 0$. But on the other hand, $(A(G_2)X)_u = (A(G_2)X)_u = 0$, a contradiction. Therefore $\lambda_1(G) < \lambda_1(G_1) + \lambda_1(G_2)$. \hfill \Box

Let $M(n, D)$ be the graph obtained from a complete graph on $n - D + 2$ vertices by removing an edge, adding a pendant path of $\lceil D/2 \rceil - 1$ edges to one end vertex of the removed edge, and adding a pendant path of $\lfloor D/2 \rfloor - 1$ edges to its other end vertex as shown in Fig. 1.
Some Results on the Largest and Least Eigenvalues of Graphs

E.R. van Dam [8] and P. Hansen and D. Stevanović [12] independently determined that the graph $M(n,D)$ attains the maximal spectral radius among all graphs on $n$ vertices with diameter $D$.

**Lemma 3.4 ([8], [12]).** Let $n$ and $D$ be integers with $1 < D < n$. Then the graph $M(n,D)$ is the unique graph with maximal spectral radius among all graphs on $n$ vertices with diameter $D$.

**Theorem 3.5 (Conjecture 4.2, [4]).** Let $G$ be a connected graph with diameter $D$. Then

$$\lambda_1(G) + D \leq n - 1 + 2 \cos \frac{\pi}{n + 1}$$

and the equality holds if and only if $G \cong P_n$.

**Proof.** Let $M = M(n,D)$ be shown in Fig. 1. If $G \neq M$, then by Lemma 3.4, $\lambda_1(G) < \lambda_1(M)$. Thus, $\lambda_1(G) + D < \lambda_1(M) + D$. Therefore in the following, it is sufficient to show that $\lambda_1(M) + D \leq n - 1 + 2 \cos \frac{\pi}{n + 1}$, with equality if and only if $G \cong P_n$.

Let $M' = M[V(M)\{v_0, \ldots, v_{\lceil \frac{D}{2} \rceil - 2}, v_{\lceil \frac{D}{2} \rceil + 2}, \ldots, v_D\}]$. Then it is easy to see that $M' = K_{n-D} - v_{\lceil \frac{D}{2} \rceil - 1}v_{\lceil \frac{D}{2} \rceil + 1}$. Therefore, $M'$ contains a Hamiltonian path with end vertices $v_{\lceil \frac{D}{2} \rceil - 1}, v_{\lceil \frac{D}{2} \rceil + 1}$, say $v_{\lceil \frac{D}{2} \rceil - 1}P_i v_{\lceil \frac{D}{2} \rceil + 1}$. Then $P = v_0v_1 \cdots v_{\lceil \frac{D}{2} \rceil - 1}P_i v_{\lceil \frac{D}{2} \rceil + 1} \cdots v_D$ is a Hamiltonian path of $M(n,D)$. Let $M_1 = P$ and $M_2 = M\{E(M_1)\}$. It is clear that $M_2$ consists of a complete graph of order $n - D + 1$ deleting an Hamiltonian cycle and $D - 1$ isolated vertices. Then $\lambda_1(M_1) = 2 \cos \frac{\pi}{n + 1}$ and $\lambda_1(M_2) = n - D - 1$. Then by Lemma 3.2

$$\lambda_1(M) \leq \lambda_1(M_1) + \lambda_1(M_2) = n - D + 1 + 2 \cos \frac{\pi}{n + 1}. \tag{3.1}$$

Therefore, $\lambda_1(M) + D \leq n - 1 + 2 \cos \frac{\pi}{n + 1}$.

If $G \cong P_n$, then $D = n - 1$ and $\lambda_1(G) = 2 \cos \frac{\pi}{n + 1}$. Therefore the equality holds.

For the converse, we may assume that $\lambda_1(G) + D \leq n - 1 + 2 \cos \frac{\pi}{n + 1}$. Then $G \cong M$ by Lemma 3.2 and the inequality (3.1) is equality. If $M_2 \neq \emptyset$, then $2 \leq D \leq n - 2$. Since $M_1 \cong P_n$ and $M_2$ consists of a complete graph of order $n - D + 1$ deleting a
Hamiltonian cycle and $D - 1$ isolated vertices. Therefore, by Corollary 3.3, 
$$\lambda_1(M) < \lambda_1(M_1) + \lambda_1(M_2) = n - D - 1 + 2\cos \frac{\pi}{n + 1},$$
that is, $\lambda_1(M) + D < n - 1 + 2\cos \frac{\pi}{n + 1}$, a contradiction. Then $M_2 = \emptyset$ and $M = M_1 \cong P_n$. Thus, we complete the proof. \qed

**Corollary 3.6.** Let $G$ be a connected graph with diameter $D$. Then
$$\lambda_1(G) \leq n - D - 1 + 2\cos \frac{\pi}{n + 1}$$
and the equality holds if and only if $G \cong P_n$.

**Lemma 3.7** ([11]). Let $G_{n,\mu}$ be the set of graphs on $n$ vertices with matching number $\mu$. For any $G \in G_{n,\mu}$, we have

(i) If $n = 2\mu$ or $n = 2\mu + 1$, then $\rho(G) \leq \rho(K_n)$ with equality if and only if $G \cong K_n$.

(ii) If $2\mu + 2 \leq n < 3\mu + 2$, then $\rho(G) \leq 2\mu$ with equality if and only if $G \cong K_{2\mu+1} \cup K_{n-2\mu-1}$.

(iii) If $n = 3\mu + 2$, then $\rho(G) \leq 2\mu$ with equality if and only if $G \cong K_\mu \cup K_{n-\mu}$ or $G \cong K_{2\mu+1} \cup K_{n-2\mu-1}$.

(iv) If $n > 3\mu + 2$, then $\rho(G) \leq \frac{1}{2}(\mu - 1 + \sqrt{(\mu - 1)^2 + 4\mu(n - \mu)})$ with equality if and only if $G \cong K_\mu \cup K_{n-\mu}$.

**Theorem 3.8.** Let $G$ be a graph on $n \geq 3$ vertices with spectral radius $\lambda_1(G)$ and matching number $\mu$. Then
$$\lambda_1(G) - \mu \leq n - 1 - \lfloor n/2 \rfloor$$
and the equality holds if and only if $G \cong K_n$ or $G \cong K_{n-1} \cup K_1$ and $n$ is even.

**Proof.** Let $G$ be a graph with matching number $\mu$. Then $\mu \leq \lfloor n/2 \rfloor$, thus we distinguish the following three cases.

**Case 1.** $\mu = \lfloor n/2 \rfloor$.

Then,
$$\lambda_1(G) - \mu = \lambda_1(G) - \lfloor n/2 \rfloor \leq n - 1 - \lfloor n/2 \rfloor. \tag{3.2}$$

**Case 2.** $\frac{n-2}{3} \leq \mu \leq n/2 - 1$.

Then by Lemma 3.7 (ii) and (iii), $\lambda_1(G) \leq 2\mu$. Therefore, we obtain
$$\lambda_1(G) - \mu \leq n/2 - 1 \leq n - 1 - \lfloor n/2 \rfloor. \tag{3.3}$$
Some Results on the Largest and Least Eigenvalues of Graphs

Case 3. \( \mu \leq \frac{n}{3} - 1 \).

Similar to the proof in [20], \( \lambda_1(G) - \mu < n - 1 - \floor{n/2} \).

If \( G \cong K_n \), then \( \lambda_1(G) = n - 1 \) and \( \mu = \floor{n/2} \), thus the equality holds. If \( G \cong K_{n-1} \cup K_1 \) and \( n \) is even, then \( \lambda_1(G) = n - 2 \) and \( \mu = n/2 - 1 \), thus the equality holds.

For the converse, we may suppose that \( \lambda_1 - \mu = n - 1 - \floor{n/2} \). Then all the inequalities in (3.2) and (3.3) are equalities. Since the inequality (3.2) is equality, \( \lambda_1(G) = n - 1 \) and \( \mu = \floor{n/2} \), then \( G \cong K_n \). Since the inequality (3.3) is equality. Then \( G \cong K_{2\mu+1} \cup K_{n-2\mu+1} \) by Lemma 3.7 (ii), and \( \lambda_1(G) = 2\mu, \mu = n/2 - 1 \) and \( n \) is even. Thus, \( G \cong K_{n-1} \cup K_1 \) and \( n \) is even.

**Theorem 3.9.** Let \( G \) be a graph on \( n \geq 6 \) vertices with spectral radius \( \lambda_1(G) \) and matching number \( \mu \). Then

\[
\frac{\lambda_1(G)}{\mu} \leq \sqrt{n - 1}
\]

and equality holds if and only if \( G \cong K_{1,n-1} \).

**Proof.** If \( G \) is empty, then the result follows immediately. If \( G \) is not empty, then we have \( 1 \leq \mu \leq \floor{n/2} \). Thus, we consider the following two cases.

Case 1. \( n/3 - 1 < \mu \leq \floor{n/2} \).

Then by Lemma 3.7 (i), (ii) and (iii), \( \lambda_1(G) \leq 2\mu \), thus \( \frac{\lambda_1(G)}{\mu} \leq 2 < \sqrt{n - 1} \) since \( n \geq 6 \).

Case 2. \( 1 \leq \mu \leq n/3 - 1 \).

If \( \mu = 1 \), then \( G \cong K_3 \cup K_{n-3} \) or \( G \cong K_{1,i} \cup K_{n-i-1} \) for \( 1 \leq i \leq n - 1 \). Thus, \( \lambda_1(G) \leq \lambda_1(K_{1,n-1}) = \sqrt{n - 1} \) since \( n \geq 6 \). Equality holds if and only if \( G \cong K_{1,n-1} \).

If \( 2 \leq \mu \leq n/3 - 1 \), similar to the proof in [20], \( \frac{\lambda_1(G)}{\mu} < \sqrt{n - 1} \).

Therefore, \( \frac{\lambda_1(G)}{\mu} \leq \sqrt{n - 1} \) with equality if and only if \( G \cong K_{1,n-1} \).

**Remark 3.10.** The order \( n \geq 6 \) of the graph in Theorem 3.9 is needed. By a direct computation, when \( n = 3 \), \( \frac{\lambda_1(G)}{\mu} \leq 2 \) with equality if and only if \( G \cong K_3 \); when \( n = 4 \), \( \frac{\lambda_1(G)}{\mu} \leq 2 \) with equality if and only if \( G \cong K_3 \cup K_1 \); when \( n = 5 \), \( \frac{\lambda_1(G)}{\mu} \leq 2 \) with equality if and only if \( G \cong K_5 \) or \( K_{1,4} \) or \( K_3 \cup K_2 \).

4. The minimum least eigenvalue among all quasi-tree graphs. A graph \( G \) is called minimizing (respectively, maximizing) in a certain class of graphs if the least eigenvalue (respectively, spectral radius) of \( G \) attains the minimum (respectively,
maximum) among all graphs in the class.

**Lemma 4.1.** The graph $K_{2,n-2}$ is the unique maximizing graph among all bipartite quasi-tree graphs.

**Proof.** Let $G$ be a quasi-tree maximizing graph among all bipartite quasi-tree graphs. Let $V(G) = \{v_1, v_2, \ldots, v_n\}$ and $X = (x_1, x_2, \ldots, x_n)^t$ be the Perron vector of $G$. Assume that $G' = G - v_1$ is a tree. Let $V(G) = S \cup T$, where $S$ and $T$ are the partitions of $V(G)$ such that $S$ and $T$ are independent sets. Let $x_u = \{\max x_i | v_i \in S \setminus \{v_1\}\}$. Without loss of generality, we may assume that $\sum_{v_i \in S \setminus \{v_1\}} x_i \leq \sum_{v_i \in T \setminus \{v_1\}} x_i$. We first prove the following claim.

**Claim 1.** $v_1 \in S$ and $v_1v_i \in E(G)$ for all $v_i \in T$.

By contradiction, we suppose that $v_1 \in T$. Note that $G$ is a maximizing graph among all bipartite quasi-trees, then $v_1v_i \in E(G)$ for all $v_i \in S$. Let $G' = G - \{v_1v_i | v_i \in S\} + \{v_1v_i | v_i \in T\}$. Therefore, $G'$ is also a bipartite quasi-tree graph. But $p(G') \geq 1^t A(G')X = 1^t A(G)X = 2x_1 \sum_{v_i \in S} x_i + 2x_1 \sum_{v_i \in T} x_i \geq 1^t A(G)X = p(G)$.

If $p(G') = p(G)$, then $X$ is also the Perron vector of $G'$. Since $p(G')X = p(G)X$, thus $(A(G')x_i) (A(G)x_i)$ for $i = 1, \ldots, n$. But on the other hand, for $v_i \in S$, $(A(G)x_i) = x_1 + \sum_{v_j \in N(v_i)} x_j$, and $A(G')x_i = \sum_{v_j \in N(v_i)} x_j$, a contradiction. Hence, $p(G') > p(G)$. This contradicts the maximality of $G$.

**Claim 2.** No vertex of $S \setminus \{u\}$ in $G'$ has a neighbor with degree one.

If not, suppose that $w \in S \setminus \{u\}$ has a neighbor, say $w'$ with $d_{G'}(w') = 1$. Then let $G' = G - ww' + uu'$. Obviously, $G'$ is a bipartite quasi-tree graph. But $p(G') \geq 1^t A(G')X = 1^t A(G)X = 2x w x w' + 2x u x w' + 2x w x w' \geq 1^t A(G)X = p(G)$.

Similar to the proof of Claim 1, we have $p(G') > p(G)$, a contradiction.

**Claim 3.** No vertex of $T$ in $G'$ has a neighbor with degree one.

If not, suppose that $w \in T$ has a neighbor, say $w'$ with $d_{G'}(w') = 1$. Then we let $G' = G - ww' + uu' + v_1w'$. Obviously, $G'$ is a bipartite quasi-tree graph. But since $\sum_{v_i \in S \setminus \{v_1\}} x_i \leq \sum_{v_i \in T \setminus \{v_1\}} x_i$, we have $x_1 \geq x_2$. Then $p(G') \geq 1^t A(G')X = 1^t A(G)X = 2x w x w' + 2x u x w' + 2x w x w' > 1^t A(G)X = p(G)$, a contradiction.

**Claim 4.** The degree of the vertex of $T$ in $G'$ is one.

If not, suppose that $w \in T$ and $d_{G'}(w) \geq 2$. Without loss of generality, suppose $w_1, w_2$ are two neighbors of $w$ in $G'$. Let $P_1 = ww_1 \cdots$ and $P_2 = ww_2 \cdots$ be the
longest path passing the vertex \( w, w_1 \) and \( w, w_2 \) in \( G' \), respectively. Then the end vertices of \( P_1 \) and \( P_2 \) must be leaves of \( G' \) (since \( G' \) is a tree). But by Claims 2 and 3, the neighbor of leaves must be \( u \). Then there is a cycle in \( G' \), this contradicts that \( G' \) is a tree.

Then by Claims 1–4, \( G \cong K_{2,n-2} \). Thus, we complete the proof. \( \square \)

**Theorem 4.2.** The graph \( K_{2,n-2} \) is the unique minimizing graph among all quasi-tree graphs.

**Proof.** Let \( G \) be a minimizing graph among all quasi-tree graphs, and assume that \( G - v_1 \) is a tree. Let \( X \) be a least vector of \( G \). Denote \( V_+ = \{ v_i | x_i \geq 0 \} \), \( V_- = \{ v_i | x_i < 0 \} \). If \( v_1 \in V_+ \), then we delete all edges between \( v_1 \) and the vertex of \( V_+ \).

Similarly, if \( v_1 \in V_- \), then we delete all edges between \( v_1 \) and the vertex of \( V_- \). We get a bipartite graph, denoted by \( G_0 \). Obviously, \( G_0 \) is a quasi-tree, and \( \lambda_n(G) \geq \lambda_n(G_0) \). So it is sufficient to determine the minimizing graph among all bipartite quasi-tree graphs. If \( G \) is a bipartite graph, then \( \lambda_n(G) = -\lambda_1(G) \). Then by Lemma 4.1, the graph \( K_{2,n-2} \) is the unique minimizing graph among all bipartite quasi-tree graphs. \( \square \)

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