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Invariant curves around a parabolic fixed point at infinity

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Abstract. The stability of a fixed point of an area-preserving transformation in the plane is characterized by the invariant curves which surround it. The existence of invariant curves had been extensively studied for elliptic fixed points. Here we study the similar problem for parabolic fixed points. In particular we are interested in the case where the fixed point is at infinity.

1. *Introduction*

Consider an area-preserving mapping of the plane which has a fixed point and let λ_1, λ_2 be the eigenvalues of its Jacobian at this point. Since the mapping is area preserving, the determinant of the Jacobian is 1 and it follows that $\lambda_1\lambda_2=1$. Since the mapping is real, λ_1, λ_2 are either both real or are complex conjugates. If λ_1, λ_2 are real and not ±1, the fixed point is called *hyperbolic* and it cannot be stable. If $\lambda_i=\bar{\lambda}_2$, $|\lambda_1|=|\lambda_2|=1$ and λ_1, λ_2 are not ±1, the fixed point is called *elliptic*. This case is extensively studied and it is known that under some mild conditions the fixed point is stable and it is surrounded by closed invariant curves [**6**, **7**, **10**]. Numerical examples are found in [**2–4**]. Finally, if $\lambda_1=\lambda_2=1$, the fixed point is called *parabolic*. This case attracted less attention, probably because it is a degenerate one. This will be the subject of the present work.

We consider an area-preserving mapping

$$\begin{aligned} x_1&=x+P(x,y),\\ y_1&=y+Q(x,y), \end{aligned} \tag{1.1}$$

where $P, Q=o(r)$ as $r=(x^2+y^2)^{1/2}\to 0$. Here $(x,y)=(0,0)$ is a parabolic fixed point with eigenvalues $\lambda_1=\lambda_2=1$. Such a degenerate fixed point may be stable or non-stable. For example, the mapping

$$\begin{aligned} x_1&=x+f(y),\\ y_1&=y+g(x_1), \end{aligned}$$

is area preserving; however, if f, g vanish at 0 and are positive elsewhere, the iterated mappings of each (x_0, y_0), x_0, $y_0>0$, diverge to infinity. The present work discusses when parabolic fixed points are surrounded by closed invariant curves and exhibit stable behaviour.

The main tool to establish the existence of invariant curves is Moser's theorem about twist mappings [**6**, Theorem 3; **10**, § 32]:

THEOREM. *Given the mapping*

$$\theta_1 = \theta + \gamma\rho + f(\rho, \theta),$$
$$\rho_1 = \rho + g(\rho, \theta),$$

in the annulus $a < \rho < b$, $-\infty < \theta < \infty$, $b - a \geq 1$, *where* f, g *are real-analytic and periodic in* θ *and every closed curve sufficiently close to* $\rho =$ constant *intersects its image curve. For* $\varepsilon > 0$ *there exists* δ, *independent of* γ, *such that if*

$$|f| + |g| < \gamma\delta,$$

then the mapping admits infinitely many invariant curves of the form

$$\rho = v(t), \quad \theta = t + u(t),$$

with y, v *real-analytic, periodic functions and* $|u(t)|$, $|v(t) - \omega| < \varepsilon$.

The assumption that f, g are real analytic is not essential. In [**9**] it is only assumed that $f, g \in C^5$, and in [**5**] that $f, g \in C^{3+\alpha}$.

We shall show that under suitable assumptions the mapping (1.1) can be transformed into a twist mapping to which the theory of Moser may be applied.

Most of this work deals with parabolic fixed points at infinity. However, we begin it in § 2 with a result about a finite fixed point, which motivates the rest of the discussion. § 3 deals with a parabolic fixed point at infinity and explains how it differs from a finite fixed point. We discuss the mapping when its behaviour is determined by the first nonlinear terms and distinguish between the cases when these terms are homogeneous of order h, $h \leq -2$, and when they are of order $h = -1$. In § 4 we study a mapping which is analytic but its dominant part is only piecewise continuous at infinity.

The most interesting mappings are, of course, the area-preserving ones. However, because of reasons which will be clarified later on, we prefer to mention first a result which emphasizes the intersection property rather than the area-preserving property. For an area-preserving mapping this result is a particular case of a more general theorem of Simo [**11**] and we also use the same ideas.†

2. *Invariant curves around a finite fixed point*

THEOREM 1. *Given a real-analytic mapping*

$$x_1 = x + P(x, y), \qquad y_1 = y + Q(x, y), \tag{2.1}$$

in a neighborhood of $(0, 0)$, *which can be written as*

$$x_1 = x + p(x, y) + \hat{p}(x, y), \qquad y_1 = y + q(x, y) + \hat{q}(x, y), \tag{2.2}$$

† The authors wish to express their gratitude to the referee who drew their attention to Simo's work.

where $p(x, y)$, $q(x, y)$ are polynomials homogeneous of degree h, $h>1$, and $\hat{p}$, $\hat{q}$ are such that

$$\hat{p}(x, y), \hat{q}(x, y) = O(r^{h+1}) \quad \text{near } r = 0. \tag{2.3}$$

The fixed point $(0, 0)$ is surrounded by closed invariant curves provided the following assumptions hold:

(a) $$xq(x, y) - yp(x, y) \neq 0 \quad \text{for } (x, y) \neq (0, 0), \tag{2.4}$$

(b) $$p_x + q_y = 0, \tag{2.5}$$

(c) *Every closed curve sufficiently close to $xq - yp = \text{constant}$ intersects its image curve.*

If (2.1) *is area preserving, then assumptions* (b) *and* (c) *hold true automatically.*

Proof. First we present a full proof of this simple result since it motivates the rest of the work and then describe its relation to [**11**].

The differential system

$$\begin{aligned} x' &= p(x, y), \\ y' &= q(x, y), \end{aligned} \tag{2.6}$$

is a Hamiltonian system with

$$H(x, y) = [xq(x, y) - yp(x, y)]/(h+1). \tag{2.7}$$

Indeed, by the homogeneity of degree h of q, $xq_x + yq_y = hq$, and by (2.5),

$$(h+1)H_x = (xq_x + q) - yp_x = (xq_x + yq_y) + q = (h+1)q, \quad H_y = -p. \tag{2.8}$$

We may assume, for example, that

$$H(x, y) > 0 \quad \text{for } (x, y) \neq (0, 0).$$

Let us define

$$\begin{aligned} \rho &= H(x, y)^{(n-1)/(h+1)}, \\ \theta &= \int_0^{\operatorname{arctg}(y/x)} H(\cos\phi, \sin\phi)^{-2/(h+1)}\, d\phi, \end{aligned}$$

where the values of arctg pass from one branch to another as (x, y) surrounds the origin.

Since $H(x, y) \neq 0$ for $(x, y) \neq (0, 0)$ and $H(x, y)$ is a homogeneous polynomial, $H(x, y) = c$ are closed curves for $c > 0$. They are also starlike since

$$(d/dt) \operatorname{arctg}(y/x) = (y'x - x'y)/(x^2 + y^2) = (xq - yp)/r^2 \neq 0.$$

Thus $H(x, y) = c$ is a family of closed, starlike, similar curves which cover $R^2 \setminus \{0\}$. Now, ρ increases along every ray through $(0, 0)$ and θ increases with $\arg(y/x)$, hence the correspondence $(x, y) \leftrightarrow (\rho, \theta)$ is locally 1:1 in $R^2 \setminus \{0\}$. When one returns to a point after surrounding the origin, θ is increased by

$$\omega = \int_0^{2\pi} H(\cos\phi, \sin\phi)^{-2/(h+1)}\, d\phi,$$

so our change of variables is even globally one-to-one. ρ, θ are, of course, real analytic functions of x, y for $(x, y) \neq (0, 0)$.