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Extension Property

Let $K \subset \mathbb{R}^d$ be a compact set, $\mathcal{E}(K)$ be the space of traces on $K$ of $C^\infty$-functions defined on $\mathbb{R}^d$. $K$ has the extension property (EP), if $\exists$ a lceo $W : \mathcal{E}(K) \longrightarrow C^\infty(\mathbb{R}^d)$.

B.S.Mityagin (1961): $[-1, 1]$ has EP, $W$ is given by extending of elements of topological basis; $\{0\}$ does not have EP as $\mathcal{E}(\{0\})$ is not complemented in $C^\infty[-1, 1]$. By E.Borel, $\mathcal{E}(\{0\}) = \mathbb{R}^N$. Here every nbhd contains a line.

Problem (in our terms): What is a geometric characterization of EP?

There always exists a linear extension operator (by the axiom of choice) and a continuous extension operator (by Whitney’s extensions).

Let $K \subset I = [a, b]$ be perfect. Topology $\tau$ in $\mathcal{E}(K)$ is given by norms $\|f\|_q = |f|_{q,K} + sup \{ |(R_y^q f)^{(k)}(x)| \cdot |x-y|^{k-q} : x, y \in K, x \neq y, k \leq q\}$ for $q \in \mathbb{Z}_+$, where $|f|_{q,K} = sup \{ |f^{(k)}(x)| : x \in K, k \leq q\}$. Also let $\|f\|_q = inf \{ \|f|_{q,I} : F|_K = f\}$. Then the quotient $\tau_Q$, given by $(\|\cdot\|_q^\infty)_{q=0}$, is complete. By OMT, $\tau_Q \sim \tau$, so \( \forall q \exists r, C : \|f\|_q \leq C \|f\|_r \).

In general, $F$ that realize $\|f\|_q$ depend on $q$. EP gives simultaneous extensions.
Linear Topological Characterization of $EP$

Let $E, F$ be nuclear Fréchet spaces, $s$ be the space of rapidly decreasing sequences and a short exact sequence $0 \longrightarrow F \xrightarrow{i} s \xrightarrow{\pi} E \longrightarrow 0$ be exact. Then, by D.Vogt and others, it splits iff $F$ has $(\Omega)$ property, whereas $E$ has a dominating norm ($DN$).

Recall that a Fréchet space $X$ with an increasing system of seminorms $(\| \cdot \|_k)_{k=0}^{\infty}$ has a dominating norm $\| \cdot \|_p$ if for each $q \in \mathbb{N}$ there exist $r \in \mathbb{N}$ and $C \geq 1$ such that $\| \cdot \|_q^2 \leq C \| \cdot \|_p \| \cdot \|_r$.

Important application: $F = \mathcal{F}(K, I) := \{ F \in C^\infty(I) : F|_K = 0 \}$ is the space of flat on $K$ functions, $E = \mathcal{E}(K)$. Then for each $K$ the sequence

$$0 \longrightarrow \mathcal{F}(K, I) \xrightarrow{i} C^\infty(I) \xrightarrow{\pi} \mathcal{E}(K) \longrightarrow 0$$

is exact. If it splits then $\exists W = \pi^{-1}$, the right inverse.

By M.Tidten (1979), $\mathcal{F}(K, I)$ always has $(\Omega)$. Therefore, a compact set $K$ has EP if and only if the space $\mathcal{E}(K)$ has a dominating norm.
Markov Property, Jackson topology and EP

Let $\mathcal{P}_n$ denote the set of all holomorphic polynomials of degree at most $n$, $K \subset \mathbb{C}$ with $\#(K) = \infty$. Then the $n$–th Markov’s factor of $K$ is

$$M_n(K) = \inf \{ M : |P'|_{0,K} \leq M |P|_{0,K}, \; P \in \mathcal{P}_n \},$$

so it is the norm of the operator of differentiation in $(\mathcal{P}_n, |\cdot|_{0,K})$. We say that a set $K$ is Markov if $(M_n(K))_n$ is of polynomial growth.

W. Pawłucki and W. Pleśniak (1988) constructed, for $K$ with pol-l cusps, $\mathcal{W}$ in the form of a telescoping series containing Lagrange interpolation pol-ls with Fekete nodes. W. Pleśniak generalized this to any Markov set. Let $d_{-1}(f) = |f|_0$, $d_0(f) = E_0(f)$, $d_k(f) = \sup_{n \geq 1} n^k E_n(f)$ for $k \geq 1$. Here, $E_n(f)$ denotes the best approximation to $f$ on $K$ by $P \in \mathcal{P}_n$. The Jackson topology $\tau_J$ is not stronger than $\tau$. By W. Pleśniak (1990), $\exists \mathcal{W} : (\mathcal{E}(K), \tau_J) \rightarrow C^\infty(\mathbb{R}^d)$ is continuous $\iff \tau_J \sim \tau \iff K$ is Markov.

AG (1996, 97): $\exists$ non-Markov $K$ with EP, so the space $(\mathcal{E}(K), \tau_J)$ is not complete. Question: Is $\mathcal{W} : (\mathcal{E}(K), \tau) \rightarrow C^\infty(\mathbb{R}^d)$ bounded?
Some Geometric Conditions for $EP$

$EP$ is valid for $[-1, 1]$, closure of domain with smooth boundary, $Lip$ boundary, Hölder boundary (B.Mityagin, R.Sealey, E.Stein, M.Valdivia, D.Vogt, E.Bierstone, ...).

M.Tidten (1983): $K$ is uniformly perfect $\Rightarrow$ $EP$ $\Rightarrow$ $\exists m : K$ is $m$–perfect. The last means: $\exists C, \delta : \forall y \in K \ \exists (x_j)_{j=1}^{\infty} \subset K : |y - x_j| \downarrow 0, |y - x_1| \geq \delta$ and $C \cdot |y - x_{j+1}| \geq |y - x_j|^m, \forall j$.

A.G.& M.Kocatepe (1997): $K = \{0\} \cup \bigcup_{n=1}^{\infty} I_n$ - “one-dimensional cusp”. $K$ has $EP$ $\iff$ $\exists \varepsilon : |I_n|^\varepsilon \geq \text{dist}(I_n, I_{n-1})$.

A.G.(1997,2001): Cantor-type set $K^{(\alpha)} = \cap_{s=0}^{\infty} \cup_{j=1}^{2^s} I_{j,s}$ with $|I_{j,s}| = l_s = l_{s-1}^\alpha$. Then $K^{(\alpha)}$ has $EP$ $\iff$ $\alpha \leq 2$. Here, $K^{(\alpha)}$ is $m$–perfect $\iff$ $\alpha \leq m$.

A.G.(2007): $K^{(\alpha_s)}$ - generalization of $K^{(\alpha)}$: $|I_{j,s}| = l_s = l_{s-1}^{\alpha_s}$. Let $\pi_{n,0} = 1$ and $\pi_{n,k} = 2^{-k} \alpha_{n+1} \cdots \alpha_{n+k}$, $\sigma_{n,s} = \sum_{k=0}^{s} \pi_{n,k}$. $K^{(\alpha_n)}$ has $EP$ $\iff$ $\sigma_{n,s+1} / \sigma_{n,s} \Rightarrow 1, s \to \infty$ uniformly wrt $n$.

This condition is related to the theory of logarithmic potential.
Equilibrium Cantor Sets

Let \( \gamma = (\gamma_s)_{s=1}^{\infty} \) with \( 0 < \gamma_s < \frac{1}{4} \), \( r_0 = 1 \) and \( r_s = \gamma_s r_{s-1}^2 \) for \( s \in \mathbb{N} \). Take \( P_2(x) = x(x - 1) \) and, recursively, \( P_{2s+1} = P_{2s} \cdot (P_{2s} + r_s) \) for \( s \in \mathbb{N} \). Then \( P_{2s} \) has \( 2^{s-1} \) minima with equal values \( P_{2s} = -r_{s-1}^2/4 \). Consider the sets \( D_s = \{ z \in \mathbb{C} : |P_{2s}(z) + r_s/2| < r_s/2 \} \) (we have \( D_s \searrow \)) and \( E_s := \{ x \in \mathbb{R} : |P_{2s}(x) + r_s/2| \leq r_s/2 \} = \bigcup_{j=1}^{2^s} I_{j,s} \). Also, \( E_{s+1} \subset E_s \). Let \( K(\gamma) := \bigcap_{s=0}^{\infty} \overline{D_s} = \bigcap_{s=0}^{\infty} E_s = \bigcap_{s=0}^{\infty} \left( \frac{2}{r_s} P_{2s} + 1 \right)^{-1} \left( [-1, 1] \right) \).

Here, \( P_{2s} + r_s/2 = T_{2s,K(\gamma)} \). If \( \gamma_s = \frac{1}{4} \) for all \( s \), then \( K(\gamma) = [0, 1] \).

We have \( \text{Rob}(K(\gamma)) = \sum_{k=1}^{\infty} B_k \), where \( B_k = 2^{-k-1} \log \frac{1}{\gamma_1 \gamma_2 \cdots \gamma_k} \).

If \( \gamma_n < 1/32 \) and \( \sum_{n=1}^{\infty} \gamma_n < \infty \), then the equilibrium measure \( \mu_{K(\gamma)} = \Lambda_{\text{h}} \) – the corresponding Hausdorff measure.

A.G. & Z.Ural (2016): Suppose \( \gamma_n < 1/32 \) and \( \sum_{n=1}^{\infty} \gamma_n < \infty \). Then \( K(\gamma) \) has \( EP \iff B_{n+s}/\sum_{k=n}^{n+s} B_k \to 0 \) as \( s \to \infty \), uniformly with respect to \( n \). Here, \( K(\gamma) \) may be polar.
Hausdorff measures and Hausdorff contents

Let $h$ be a measure function, that is a continuous on $[0, \infty)$ nondecreasing function with $h(0) = 0$. Given set $K \subset (X, d)$, its $h$–Hausdorff content is $M_h(K) = \inf \{ \sum h(r_j) : K \subset \bigcup B(x_j, r_j) \}$ and $h$–Hausdorff measure is $\Lambda_h(K) = \lim_{\delta \to 0} \inf \{ \sum h(r_j) : K \subset \bigcup B(x_j, r_j) \text{ with } r_j \leq \delta \}$.

Logarithmic measure is given by $h_0(r) = (\log \frac{1}{r})^{-1}$, $0 < r < 1$.

Let $K \subset \mathbb{C}$. By J.Lindeberg (1918), $\Lambda_{h_0}(K) = 0$ implies that $K$ is exceptional ($\text{Cap}(K) = 0$). On the other hand, by R.Nevanlinna (1930ths), $\Lambda_h(K) > 0 \Rightarrow \text{Cap}(K) > 0$ for each $h$ with $\int_0^\infty \frac{h}{r} \, dr < \infty$. But, by H.Ursell (1938), between $h$ with $\int_0^\infty \frac{h}{r} \, dr < \infty$ and $h_0$ there is a zone of uncertainty. Namely, $\exists h_1, h_2$ with $h_2 = o(h_1)$, $\int_0^\infty \frac{h_i}{r} \, dr = \infty$, non-polar $K_1$ and polar $K_2$, such that $\Lambda_{h_2}(K_2) > 0$, whereas $\Lambda_{h_1}(K_1) < \infty$, so $\Lambda_{h_2}(K_1) = 0$. This means that there is no general characterization of polarity of sets in terms of Hausdorff measures. Similarly, we can show that

there are $h_2 \prec h_1$ and $K_1, K_2$, where $K_j$ is an $h_j$–set ($0 < \Lambda_{h_j}(K_j) < \infty$): the smaller set $K_1$ has EP, whereas the larger set $K_2$ does not have.
EP and densities of Hausdorff contents

Clearly, EP is a local property, it is valid if a set is not “very small” near each of its point. One can suggest to characterize EP in terms of lower densities of Hausdorff contents of sets.

Given \( h \succ t \), a set \( K \) and \( r > 0 \), let \( \varphi_{h,K}(r) := \inf_{x \in K} M_h(K \cap B(x, r)) \), with \( B(x, r) = [x - r, x + r] \). Then the lower density of \( M_h \)
\[
\phi_h(K) := \liminf_{r \to 0} \inf_{x \in K} \frac{M_h(K \cap B(x, r))}{M_h(B(x, r))}
\]
is equal to \( \liminf_{r \to 0} \frac{\varphi_{h,K}(r)}{h(2r)} \).

In order to distinguish EP by means of \( \phi_h \), we have to consider \( h \succ h_0 \), since there are sets \( K \) with EP and \( \Lambda_{h_0}(K) = 0 \), so \( M_{h_0}(K) = 0 \).

We analyze a wide class of dimension functions and show that lower densities of Hausdorff contents do not distinguish EP. Namely, let
\[ h = h_0^\alpha \] with \( \alpha(t) = \alpha_0 - \varepsilon(t) \) for \( 0 < \alpha_0 \leq 1 \) and regular \( \varepsilon \searrow 0 \).

Then for “one-dimensional cusp” \( K \) we have EP \iff \( \phi_h(K) > 0 \).

On the other hand, there are \( K(\gamma) \) without EP and positive value of \( \phi_h \).
Recall that if $M_n(K) \leq n^Q$ for some $Q$, then, by W. Pleśniak, $K$ has EP. 

One can suppose that there is a maximal growth of Markov’s factors that separates sets with EP from sets without it.

Unfortunately - not! There is no criterion of EP in terms of growth of Markov's factors. Here, also, a zone of uncertainty is not empty. Namely, there are two sets $K_1, K_2$ such that $K_1$ has EP, $K_2$ - not, but $M_n(K_2)/M_n(K_1) \to 0$ as $n \to \infty$.

Both sets are of the type $K(\gamma)$, for which we can control the Green function and, therefore, the growth of Markov’s factors.

Existence of zones of uncertainty (for the extension property) in the scale of dimension functions and in the scale of growth rate of Markov’s factors implicates the problem to find the boundaries of these zones.
THANK YOU!