A finiteness principle for the smooth selection problem

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Outline

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We will discuss finiteness principles associated to selection problems.

The first problem has to do with the smooth selection of function values of a vector-valued function (SSP).

The second problem has to do with the smooth selection of jets of a scalar-valued function (JSP).
Preliminary notation:

$C^{m}(\mathbb{R}^{n})$ denotes the space of real-valued $C^{m}$ functions on $\mathbb{R}^{n}$.

$\mathcal{P}$ denotes the space of $(m-1)'st$ degree polynomials on $\mathbb{R}^{n}$.

We write $J_{x}(F)$ to denote the $(m-1)$-jet at $x$ of a function $F$. Hence, $J_{x}(F) \in \mathcal{P}$

$C^{m}(\mathbb{R}^{n} \rightarrow \mathbb{R}^{D})$ denotes the space of $\mathbb{R}^{D}$-valued $C^{m}$ functions on $\mathbb{R}^{n}$, namely the space of all functions $F = (F_{1}, \cdots, F_{D})$, where $F_{i} \in C^{m}(\mathbb{R}^{n})$ for each $i$.

Fix $m \geq 0$, $n \geq 1$, and $D \geq 1$.

Constants of the form $C$, $l^{*}$, $k^{\#}$ will usually depend only on $m, n, D$. 
The smooth selection problem.

Fix:

- A cube $Q_0 \subset \mathbb{R}^n$ of sidelength $\delta_0$.
- A finite set $E \subset Q_0$. 
Smooth Selection Problem:

Given a collection of convex sets $K(x) \subset \mathbb{R}^D$ for each $x \in E$, and given $M > 0$, we want to know whether there is a function $F \in C^m(\mathbb{R}^n \to \mathbb{R}^D)$ such that

$$
\begin{cases}
F(x) \in K(x) \quad \forall x \in E \\
\|F\|_{C^m} \leq M.
\end{cases}
$$

We call $F$ a $C^m$-selection of $\vec{K} = (K(x))_{x \in E}$, and we say that $F$ is a solution of $SSP(\vec{K}, M)$. 
Our goal is to determine a number $M_0 \in [0, \infty)$ such that

- $SSP(\vec{K}, M)$ has a solution if $M > CM_0$, but
- $SSP(\vec{K}, M)$ does not have a solution if $M < C^{-1}M_0$,

for a constant $C > 1$ depending only on $m, n, D$. 
Finiteness principle for the Smooth Selection Problem

**Theorem A**
There are constants $k^\# = k^\#(m, n, D)$ and $C = C(m, n, D)$ such that the following holds.

Assume the **Finiteness Hypothesis**: For each $S \subset E$ with $\#(S) \leq k^\#$ there exists $F^S \in C^m(\mathbb{R}^n \to \mathbb{R}^D)$ with norm $\|F^S\|_{C^m} \leq M_0$ such that $F^S(x) \in K(x)$ for all $x \in S$.

Then there exists $F \in C^m(\mathbb{R}^n \to \mathbb{R}^D)$ with norm $\|F\|_{C^m} \leq CM_0$ such that $F(x) \in K(x)$ for all $x \in E$.

**Remark:** We answer our previous question by determining the order of magnitude of the smallest possible $M_0$ for which the **Finiteness Hypothesis** holds. This can be computed using convex programming.
The rôle of convexity

Convexity of the constraint sets $K(x)$ is required for our arguments to work. We intend to define a selection of the form

$$F = \sum_{Q \in CZ} \theta_Q^2 F_Q \text{ on } 5Q_0.$$  

where $CZ$ is a partition of the cube $5Q_0$ into dyadic subcubes, $\{\theta_Q^2\}$ is a partition of unity subordinate to $\{\frac{65}{64}Q : Q \in CZ\}$, and the $F_Q$ solve local selection problems on $5Q$. In particular we know that $F_Q(x) \in K(x)$ for all $x \in E \cap \frac{65}{64}Q$.

We hope to prove that $F$ is a selection. Now, if $x \in E$, then $F(x)$ is a convex combination of vectors in $K(x)$. In order to ensure that $F(x) \in K(x)$ we must require that $K(x)$ is convex.
The jet selection problem

Definition
A family of sets $\vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0}$ is called a shape field if

1. Each $\Gamma(x, M)$ is a convex subset of $\mathcal{P}$.
2. $\Gamma(x, M) \subset \Gamma(x, M')$ for $M' > M$ and $x \in E$.
3. $|\partial^\beta P(x)| \leq M$ for all $P \in \Gamma(x, M)$. 
Jet Selection Problem: Given $\vec{\Gamma}$ and given $M > 0$, we want to know whether there is an $F \in C^m(\mathbb{R}^n)$ such that

$$\begin{cases}
J_x F(x) \in \Gamma(x, M) \quad \forall x \in E \\
\|F\|_{C^m(\mathbb{R}^n)} \leq M.
\end{cases}$$

We call $F$ a $C^m$-selection of $\vec{\Gamma}$, and we say that $F$ is a solution of $JSP(\vec{\Gamma}, M)$.

- If the jet selection problem has a solution for some $M$ then it has a solution for all $M' > M$.

- If $\exists M^*$ so that $\Gamma(x, M^*) \neq \emptyset$ for all $x \in E$, then $JSP(\vec{\Gamma}, M)$ has a solution for some $M > M^*$. 
Finiteness principle for the Jet Selection Problem

**Theorem B**
There exists a constant \( k^\# = k^\#(m, n) \) such that the following holds. Fix a finite set \( E \subset Q_0 \), where \( Q_0 \) is a cube of sidelength \( \delta_0 \). Assume the shape field \( \vec{\Gamma} = (\Gamma(x, M))_{x \in E, M > 0} \) is \((C_0, \delta_0)\)-convex (see next slide).

Assume the **Finiteness Hypothesis**: For each \( S \subset E \) with \( \#(S) \leq k^\# \) there exists \( F^S \in C^m(\mathbb{R}^n) \) with \( \|F^S\|_{C^m} \leq M_0 \) and \( J_x F^S \in \Gamma(x, M_0) \) for all \( x \in S \).

Then there exists \( F \in C^m(\mathbb{R}^n) \) with \( \|F\| \leq CM_0 \) and \( J_x F \in \Gamma(x, C M_0) \) for all \( x \in E \).

Here, \( C \) depends on \( m, n, \) and \( C_0 \)
**Definition:** We say that $\tilde{\Gamma}$ is $(C_0, \delta_0)$-convex if the following holds:

Suppose $0 < \delta \leq \delta_0$, $x \in E$, $M > 0$, $P_1, P_2 \in \Gamma(x, M)$, $R_1, R_2 \in \mathcal{P}$. Assume the estimates

\[ |\partial^\beta (P_1 - P_2)(x)| \leq M \delta^{m-|\beta|} \]
\[ |\partial^\beta R_1(x)| \leq \delta^{-|\beta|} \]
\[ |\partial^\beta R_2(x)| \leq \delta^{-|\beta|}, \ \forall \ |\beta| \leq m - 1. \]

Assume also that $R_1 \circ_x R_1 + R_2 \circ_x R_2 = 1$, where $\circ_x$ is the product on $\mathcal{P}$ defined by $P \circ_x Q = J_x(P \cdot Q)$. Then $R_1 \circ_x R_1 \circ_x P_1 + R_2 \circ_x R_2 \circ_x P_2 \in \Gamma(x, C_0 M)$.
Remarks on \((C_0, \delta_0)\)-convexity: The \((C_0, \delta_0)\)-convexity condition arises because we intend to construct a selection of the form \(F = \sum_{Q \in CZ} \theta_Q^2 F_Q\). The partition of unity \(\{\theta_Q^2\}\) is subordinate to \(\{\frac{65}{64} Q : Q \in CZ\}\) and satisfies estimates of the form \(|\partial^\alpha \theta_Q^2| \leq C \delta_Q^{-|\alpha|}\). In our setup, the \(F_Q\) will solve local extension problems on \(5Q\); in particular, \(J_x F_Q \in \Gamma(x, M)\) for all \(x \in E \cap \frac{65}{64} Q\). Also, \(F_Q - F_Q'\) will be small in a suitable sense, when \(Q, Q'\) are neighboring CZ cubes.

When we measure the jet of \(F\) at a point \(x \in E\) we will have to control sums of the form

\[
J_x F = \sum_{i=1}^{L} R_i \circ_x R_i \circ_x P_i,
\]

where \(\{P_i\}\) are jets of the form \(J_x F_{Q_i}\) and \(R_i\) are jets of the form \(J_x \theta_{Q_i}\) for a list of CZ cubes \(\{Q_i\}_{i=1}^{L}\). By bootstrapping the \((C_0, \delta_0)\)-convexity condition, we can ensure that \(J_x F\) will belong to \(\Gamma(x, CM)\) for a constant \(C\).
The relationship between Theorem A and Theorem B.

We can use Theorem B (FP for JSPs) to prove Theorem A (FP for SSPs).

We will explain how to encode a smooth selection problem for the function space \( C^m(\mathbb{R}^n \rightarrow \mathbb{R}^D) \) as a jet selection problem for the function space \( C^{m+1}(\mathbb{R}^{n+D}) \).
We will use coordinates \((x, \xi)\) on \(\mathbb{R}^{n+D} = \mathbb{R}^n \times \mathbb{R}^D\).

Let \(\mathcal{P}^+ := \) the space of real-valued \(m\)'th degree polynomials on \(\mathbb{R}^{n+D}\).

Given a smooth function \(F : \mathbb{R}^{n+D} \to \mathbb{R}\), denote

\[
\nabla_x F = (\partial_{x_1} F, \cdots, \partial_{x_n} F), \\
\nabla_\xi F = (\partial_{\xi_1} F, \cdots, \partial_{\xi_D} F),
\]

Write \(J^+_x F\) to denote the \(m\)-jet of a function \(F : \mathbb{R}^{n+D} \to \mathbb{R}\). Hence, \(J^+_x F \in \mathcal{P}^+\).
The embedding trick

Fix \( \vec{K} = (K(x))_{x \in E} \), where \( K(x) \) are convex subsets of \( \mathbb{R}^D \).

We associate to \( E \subset \mathbb{R}^n \) the set \( E^+ \subset \mathbb{R}^{n+D} \) defined by \( E^+ := \{(x, 0) : x \in E\} \).

Using \( \vec{K} \), we define a particular shape field \( \vec{\Gamma} \) on \( E^+ \).

For \( (x, 0) \in E^+ \) and \( M > 0 \), we set

\[
\Gamma((x, 0), M) = \left\{ P \in \mathcal{P}^+ : P(x, 0) = 0, \quad \nabla_\xi P(x, 0) \in K(x), \quad |\partial_x^\alpha \partial_\xi^\beta P(x, 0)| \leq M \right\}.
\]
We relate the existence of an \( F = (F_1, \cdots, F_D) \in C^m(\mathbb{R}^n \to \mathbb{R}^D) \) satisfying

\[
SSP(\vec{K}, M) : \begin{cases} 
F(x) \in K(x) \text{ for all } x \in E. \\
norm{F}_{C^m} \leq M.
\end{cases}
\]

to the existence of an \( F^+ \in C^{m+1}(\mathbb{R}^{n+D} \to \mathbb{R}) \) satisfying

\[
JSP(\vec{\Gamma}, M) : \begin{cases} 
J^+_{(x,0)} F^+ \in \Gamma((x, 0), M) \text{ for all } (x, 0) \in E+. \\
norm{F^+}_{C^{m+1}} \leq M.
\end{cases}
\]

**Embedding lemma**

If \( F^+ \) solves \( JSP(\vec{\Gamma}, M) \) then there exists an \( F \) solving \( SSP(\vec{K}, M) \).

If \( F \) solves \( SSP(\vec{K}, M) \) then there exists an \( F^+ \) solving \( JSP(\vec{\Gamma}, C_{EM}) \).
Proof of the embedding lemma

JSP $\implies$ SSP:
Suppose $F^+ \in C^{m+1}(\mathbb{R}^{n+D})$ is a solution to $JSP(\vec{\Gamma}, M)$.
Set $F(x) = \nabla_\xi F^+(x, 0)$ which is evidently a $C^m(\mathbb{R}^n \to \mathbb{R}^D)$ function.

Furthermore,
\[
\begin{aligned}
\|F\|_{C^m} &\leq \|F^+\|_{C^{m+1}} \leq M. \\
F(x) &\in K(x) \quad \forall x \in E.
\end{aligned}
\]

Hence, $F$ is a solution to $SSP(\vec{K}, M)$. 
SSP \implies JSP:

Suppose \( F = (F_1, \cdots, F_D) \in C^m(\mathbb{R}^n \to \mathbb{R}^D) \) is a solution to \( SSP(\vec{K}, M) \).
That is, \( F(x) \in K(x) \) for all \( x \in E \) and \( \|F\|_{C^m} \leq M \).

**Claim:** There exists a function \( F^+ \in C^{m+1}(\mathbb{R}^{n+D}) \) such that
\[
\|F^+\|_{C^{m+1}} \leq CM \quad \text{and} \quad F^+|_{\xi=0} = 0 \quad \text{and} \quad \nabla_\xi F^+|_{\xi=0} = F.
\]

With this claim, we can finish the proof of the embedding lemma. Indeed, for any \( (x, 0) \in E^+ \) we have \( F^+(x, 0) = 0, \nabla_\xi F^+(x, 0) = F(x, 0) \in K(x) \), and
\[
|\partial^\alpha_x \partial^\beta_\xi F^+(x, 0)| \leq \|F^+\|_{C^{m+1}} \leq CM \quad \text{for} \quad |\alpha| + |\beta| \leq m.
\]
So, \( F^+ \) is a solution to \( JSP(\vec{\Gamma}, CM) \).
Proof of claim: Given \((x_0, 0) \in \mathbb{R}^n \times \{0\}\), set

\[
P_{(x_0,0)}(x, \xi) := \xi \cdot J_{x_0} F(x) = \sum_{i=1}^{D} \xi_i J_{x_0} F_i(x)
\]

Note that

\[
P_{(x_0,0)}(x_0, 0) = \sum_{i=1}^{D} \xi_i J_{x_0} F_i(x_0)|_{\xi=0} = 0,
\]

\[
\nabla_{\xi} P_{(x_0,0)}(x_0, 0) = (J_{x_0} F_1(x_0), \cdots, J_{x_0} F_D(x_0)) = F(x_0) \in K(x_0),
\]

while

\[
|\partial_x^\alpha \partial_{\xi}^\beta P_{(x_0,0)}(x_0, 0)| \leq CM \text{ for } |\alpha| + |\beta| \leq m.
\]

Hence, \(P_{(x_0,0)} \in \Gamma(x_0, CM)\).
Proof of claim (continued): Finally, it is straightforward to show that the Whitney field \( \{P(x_0,0)\} \) \( (x_0,0) \in \mathbb{R}^n \times \{0\} \) satisfies the Whitney conditions for \( C^{m+1}(\mathbb{R}^{n+D}) \). Thus, we can use the classical Whitney extension theorem to extend the Whitney field \( \{P(x_0,0)\} \) \( (x_0,0) \in \mathbb{R}^n \times \{0\} \) to a function \( F^+ \in C^{m+1}(\mathbb{R}^{n+D} \rightarrow \mathbb{R}) \).
Lemma
The set \( \vec{\Gamma} \) is \((C_0, \delta_0)\)-convex, for a constant \( C_0 = C_0(m, n, D) \). Recall \( \delta_0 \) is the sidelength of \( Q_0 \).
Proof of Lemma: Under the conditions in the statement of \((C_0, \delta_0)\)-convexity, in particular \(R_1 \circ_x R_1 + R_2 \circ_x R_2 = 1\) and \(P_i \in \Gamma((x, 0), M)\), we have

\[
(R_1 \circ_x R_1 \circ_x P_1 + R_2 \circ_x R_2 \circ_x P_2)(x, 0) = 0
\]

and

\[
\nabla_\xi(R_1 \circ_x R_1 \circ_x P_1 + R_2 \circ_x R_2 \circ_x P_2)(x, 0)
\]

\[
= (R_1 \circ_x R_1 \circ_x \nabla_\xi P_1 + R_2 \circ_x R_2 \circ_x \nabla_\xi P_2)(x, 0) \in K(x)
\]

because \(R_1 \circ_x R_1 + R_2 \circ_x R_2 = 1\) and \(\nabla_\xi P_i(x, 0) \in K(x)\), and \(K(x)\) is convex, and furthermore

\[
|\partial^\alpha_x \partial^\beta_\xi (R_1 \circ_x R_1 \circ_x P_1 + R_2 \circ_x R_2 \circ_x P_2)(x, 0)|
\]

\[
= |\partial^\alpha_x \partial^\beta_\xi (P_2 + R_1 \circ_x R_1 \circ_x (P_1 - P_2))(x, 0)|
\]

\[
\leq CM.
\]

and hence, we’ve shown \(R_1 \circ_x R_1 \circ_x P_1 + R_2 \circ_x R_2 \circ_x P_2 \in \Gamma((x, 0), CM)\).
Proof that Theorem B implies Theorem A

We let $k^\#$ be the constant from the statement of Theorem B for the function space $C^{m+1}(\mathbb{R}^{n+D})$.

Then we have the chain of implications:

- **Finiteness Hypothesis with** $k^\#$ for $(\vec{K}, M) \implies$ (via embedding lemma)
- **Finiteness Hypothesis with** $k^\#$ for $(\vec{\Gamma}, CEM) \implies$ (via Theorem B)
- Existence of solution to $JSP(\vec{\Gamma}, CEM) \implies$ (via embedding lemma)
- Existence of solution to $SSP(\vec{K}, CEM)$. 
Proof of the finiteness principle for JSPs

Recall the setting: We fix integer $m \geq 0$ and $n \geq 1$, and

- A cube $Q_0 \subset \mathbb{R}^n$ of sidelength $\delta_0$.
- A finite set $E \subset Q_0$.
- A $(C_0, \delta_0)$-convex shape field $\vec{\Gamma}_0 = (\Gamma_0(x, M))_{x \in E, M > 0}$, where $\Gamma_0(x, M)$ is a convex set of polynomials of degree at most $m - 1$ on $\mathbb{R}^n$.
- A sufficiently large integer constant $k^\#$. 
We want to know whether, for a given $M > 0$, there is a solution to

$$JSP(\vec{\Gamma}, M) : \begin{cases} J_x F \in \Gamma_0(x, M) \quad \forall x \in E \\ \|F\|_{C^m} \leq M. \end{cases}$$

(1)

Our procedure for answering this question consists in building a set of necessary conditions for the existence of a selection $F$ for the given $M > 0$. We recognize a trivial necessary condition:

$$\text{(NC}_0) \quad \Gamma_0(x, M) \neq \emptyset \quad \forall x \in E.$$
The gameplan:

- We will describe a set of conditions \((NC_1), (NC_2), \cdots, (NC_{l^*})\) which are necessary for the existence of a \(C^m\) selection. These conditions arise by iteratively applying Taylor's theorem to obtain new conditions on the jets of a selection.

- We will prove a stabilization lemma that says that for sufficiently large \(l^* = l^*(m, n)\), the conditions \((NC_1), (NC_2), \cdots, (NC_{l^*})\) imply the existence of an \(C^m\) selection.

- We will then show that the finiteness hypothesis implies the conditions \((NC_1), (NC_2), \cdots, (NC_{l^*})\) as long as \(k^\#\) is large enough.
The refinement procedure.

Our goal is to define a descending sequence of convex sets

\[ \Gamma_0(x, M) \supset \Gamma_1(x, M) \supset \cdots \supset \Gamma_l(x, M) \supset \cdots \]

satisfying the following properties:

1. \( \overrightarrow{\Gamma}_l = (\Gamma_l(x, M))_{x \in E, M > 0} \) is a \((C_l, \delta_0)\)-convex shape field for all \( l \geq 0 \), for a constant \( C_l \) determined by \( C_0, m, n, \) and \( l \).

2. If \( F \in C^m(\mathbb{R}^n) \) satisfies \( J_x F \in \Gamma_0(x, M) \) for all \( x \in E \), and \( \|F\|_{C^m} \leq M \), then \( J_x F \in \Gamma_l(x, M) \) for all \( x \in E \) and all \( l \geq 0 \). Hence, in particular, the condition "\( \Gamma_l(x, M) \neq \emptyset \ \forall x \in E \)" is necessary for the existence of a selection.
To accomplish our goal, given \( l \geq 0 \), we define inductively

\[
\Gamma_{l+1}(x, M) := \left\{ P \in \Gamma_l(x, M) : \text{ for all } y \in E, \right. \\
\left. \text{there exists } P' \in \Gamma_l(y, M) \text{ s.t.} \right. \\
|\partial^\beta (P - P')(x)| \leq C_T M |x - y|^{m-|\beta|} \text{ for } |\beta| \leq m \}. 
\]

where \( C_T \) is the constant arising in Taylor’s theorem for \( C^m(\mathbb{R}^n) \) functions.

We call \( \Gamma_l \) the “\( l \)'th refinement of \( \Gamma_0 \)”. 

We prove the properties of the first refinement $\vec{\Gamma}_1 = (\Gamma_1(x, M))_{x \in E, M > 0}$.

A straightforward calculation shows that $\vec{\Gamma}_1$ is a $(C_1, \delta_0)$-convex shape field for a constant $C_1$ depending on $m$, $n$, and $C_0$.

Assume $F \in C^m(\mathbb{R}^n)$ satisfies $\|F\|_{C^m} \leq M$ and $J_x F \in \Gamma(x, M)$ for all $x \in E$. Then, for any $x, y \in E$, we have by Taylor’s theorem that

$$|\partial^\beta (J_x F - J_y F)(x)| \leq C_T M |x - y|^{m-|\beta|}.$$ 

Now, since $J_y F \in \Gamma_0(y, M)$, we conclude that $J_x F \in \Gamma_1(y, M)$. Hence, by iterating this argument, we conclude that if $\|F\|_{C^m} \leq M$ then

$$J_x F \in \Gamma_0(x, M) \forall x \implies J_x F \in \Gamma_1(x, M) \forall x \implies J_x F \in \Gamma_2(x, M) \forall x \implies \cdots$$

and all the shape fields $\vec{\Gamma}_l$ are $(C_l, \delta_0)$-convex.
Two ingredients in the argument

Our finiteness principle for JSPs will then follow from the next two results

**Finiteness lemma**
Given \( l \geq 1 \), there exists \( k^{\#} = k^{\#}(l, m, n) \) such that the following holds. Suppose that for all \( S \subset E \) with \( \#(S) \leq k^{\#} \) there exists an \( F^S \in C^m(\mathbb{R}^n) \) such that \( \|F^S\|_{C^m} \leq M \) and \( J_xF^S \in \Gamma(x, M) \) for all \( x \in E \). Then \( \Gamma_l(x, CM) \neq \emptyset \) for all \( x \in E \).
In fact, one can take \( k^{\#} = (\dim \mathcal{P} + 2)^l \).

**Stabilization theorem (weak version)**
There is a constant \( l_* = l_*(m, n) \) such that if \( \Gamma_{l^*}(x_0, M_0) \neq \emptyset \) for some \( x_0 \in E, M_0 > 0 \), then there exists an \( F \in C^m(\mathbb{R}^n) \) with \( \|F\|_{C^m} \leq CM_0 \) and \( J_xF \in \Gamma(x, CM_0) \) for all \( x \in E \).
**Proof of finiteness lemma**

Consider the case \( l = 1 \). Then \( k^\# = (\dim \mathcal{P} + 2)^l = \dim \mathcal{P} + 2 \). Note that \( \Gamma_1(x, M) \) can be written as the intersection over all \( y \in E \) of convex subsets

\[
K_y(x) := \left\{ P \in \Gamma_0(x, M) : \exists P' \in \Gamma_0(y, M) \text{ s.t.} \right. \\
|\partial^\beta (P - P')(x)| \leq C_T M |x - y|^{m-|\beta|} \text{ for } |\beta| \leq m \left. \right\} \subset \mathcal{P}.
\]
By Helly's Theorem, this intersection is nonempty iff any \( \dim P + 1 = k^\# - 1 \) element subcollection of \( \{K_{x,y}\}_{y \in E} \) has nonempty intersection. 

Thus, we need to show that if \( y_1, \ldots, y_{k^\#-1} \in E \) then

\[
\bigcap_{i=1}^{k^\#-1} K_{x,y_i} = \left\{ P \in \Gamma_0(x, M) : \exists P_i \in \Gamma_0(y, M) \text{ s.t.} \right. \\
|\partial^\beta (P - P_i)(x)| \leq C_T M |x - y|^{m - |\beta|} \text{ for } |\beta| \leq m, 1 \leq i \leq k^\# - 1 \right\}. 
\]

is nonempty. By the **finiteness hypothesis**, there is a \( C^m \) selection \( F \) associated to the \( k^\# \) element subset \( S = \{x, y_1, \ldots, y_{k^\#-1}\} \). If we take \( P = J_x F \) and \( P_i = J_{y_i} F \), then we see that

\[
P \in \bigcap_{i=1}^{k^\#-1} K_{x,y_i}.
\]
We omit the proof of the case $l \geq 2$, which follows by iterating the previous argument.
Comments on the proof of the Stabilization theorem:

The proof of the Stabilization theorem uses the technology of labels, local selection problems, and the Calderón-Zygmund decompositions detailed in Charlie’s lectures.
In the present setting, a label $\mathcal{A}$ will consist of a set of multiindices of order at most $m - 1$.

We say that $(P_\alpha)_{\alpha \in \mathcal{A}}$ is an $(\mathcal{A}, \delta, C_B)$-basis for a convex set $\Gamma \subset P$ at $(x_0, M_0, P^0)$ if

- $P^0 \in \Gamma$.
- $P^0 + C_B^{-1} M_0 \delta^{m-|\alpha|} P_\alpha$, $P^0 - C_B^{-1} M_0 \delta^{m-|\alpha|} P_\alpha \in \Gamma$ for each $\alpha \in \mathcal{A}$.
- $\partial^\beta P_\alpha(x_0) = \delta_{\alpha\beta}$.
- $|\partial^\beta P_\alpha(x_0)| \leq C_B \delta^{\alpha - |\beta|}$. 
One then introduces the notion of a local selection problem, and defines what it means to say that a local selection problem carries a label $\mathcal{A}$. Unfortunately, we do not have enough time to cover this. Due to complications related to transporting a polynomial from one basepoint to another, an additional notion is required to allow the proof to proceed; namely, the notion of a “monotonic label” is a new detail required in this setting. One focuses only on the monotonic labels so as to obtain a crucial consistency condition.
Thanks!