Abstract

We give two simple generalizations of commutative rings.
They form (co)-complete categories, that contain commutative (semi-) rings (e.g. $\{0,1\} \subseteq [0,1] \subseteq [0,\infty)$ with the usual multiplication $x + y := \max\{x, y\}$). But they also contains the "integers" $\mathbb{Z}_{\mathbb{R}}$ (and $\mathbb{Z}_{\mathbb{C}}$), and the "residue fields" $\mathbb{F}_{\mathbb{R}}$ (and $\mathbb{F}_{\mathbb{C}}$), of the real (and complex) numbers. Here $\mathbb{Z}_{\mathbb{R}}$ is the collection of unit L_2 balls, and $\mathbb{F}_{\mathbb{R}}$ is the collection of spheres augmented with a 0. The initial object is "the field with one element" $\mathbb{F}_1$.

One generalization, $\mathcal{GR}_c$ - the "commutative generalized rings", is an axiomatization of finitely generated free modules over a commutative ring, together with the operations of multiplication and contraction. This is the more geometric language: for any $A \in \mathcal{GR}_c$ we associate its (symmetric) spectrum, $\mathrm{Spec}\, A$, a compact Zariski space, with a sheaf of $\mathcal{GR}_c$ over it. By glueing such spectra we get generalized schemes $\mathcal{G}$Sch, a full sub-category of the locally-generalized-ringed-spaces.
For a number field K, with the ring of integers $\mathcal{O}_K$, the compatification of $\mathrm{Spec}\, \mathcal{O}_K$ is a pro-object $\overline{\mathrm{Spec}\, \mathcal{O}_K} \in$ pro-$\mathcal{G}$Sch, and its points are the valuation-sub-$\mathcal{GR}_c$ of K: $\mathrm{Val}(K) \equiv \{K; \mathcal{O}_{K,\mathfrak{p}} = S_{\mathfrak{p}}^{-1}\mathcal{O}_K,\ \mathfrak{p} \subseteq \mathcal{O}_K$ finite prime $; K \cap \sigma^{-1}\mathbb{Z}_{\mathbb{C}}\ , \sigma : K \hookrightarrow \mathbb{C}\}$.
For $A \in \mathcal{GR}_c$, we have a (co)-complete abelian category of A- modules with enough injectives and projectives. For $k \to A$ in $\mathcal{GR}_c$, we obtain the A- module of Kähler differentials $\Omega(A/k)$, satisfying all the usual properties. We compute the universal derivation $d : \mathbb{Z} \to \Omega(\mathbb{Z}/\mathbb{F}_1)_{[1]}$.

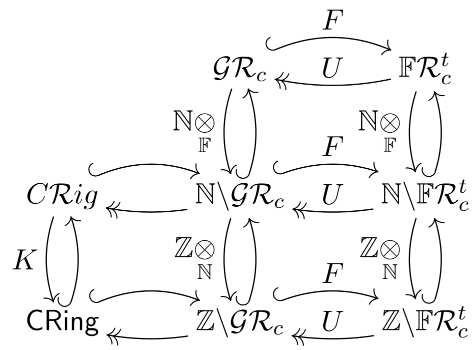
All these remain true for the second generalization $\mathbb{F}\mathcal{R}_c^t$- the "commutative $\mathbb{F}$-Rings with involution", the axiomatization of the category of finitely generated free A-modules with A-linear maps, and the operations of composition,direct sum, and taking transpose.
This is the more "linear", or K-theoretic language: for $A \in \mathbb{F}\mathcal{R}_c^t$, we have its algebraic K-theory spectum: $B(A^{-1}A) \simeq \mathbb{Z} \times BGL_\infty(A)^+$, and for $A = \mathbb{F}_1$ we obtain the sphere spectrum $\mathbb{Z} \times B\Sigma_\infty^+$.
For a compact valuation $\mathbb{F}\mathcal{R}_c^t$ we associate a "zeta" function, so that we obtain the usual factor $(1 - p^{-s})^{-1}$ for the p-adic integers $\mathbb{Z}_p$, while we get $2^{\frac{s}{2}}\Gamma(\frac{s}{2})$ for the real integers $\mathbb{Z}_{\mathbb{R}}$.

For $X = \{X_N\} \in$ pro-$\mathcal{G}$Sch, we define the category of vector bundles over X, by a certain completion of the categories of vector bundles on the finite layers X_N. For a number field K, the isomorphism classes of rank n vector bundles over $\overline{\mathrm{Spec}\, \mathcal{O}_K}$ are in natural bijection with

$$GL_n(K)\backslash GL_n(\mathbb{A}_K)/\prod_{\nu} GL_n(\mathcal{O}_{K,\nu}).$$

where $GL_n(\mathcal{O}_{K,\nu}) = \mathcal{O}(n)$ (resp. $U(n)$) for real (resp. complex) place ν of K. E.g. for $n = 1$: $\mathrm{Pic}(\overline{\mathrm{Spec}\, \mathcal{O}_K}) = K^*\backslash \mathbb{A}_K^*/\prod_\nu \mathcal{O}_{K,\nu}^*$, and for $K = \mathbb{Q}$: $\mathrm{Pic}(\overline{\mathrm{Spec}\, \mathbb{Z}}) = \mathbb{R}^+$.

We have the following "commutative" diagram of adjunctions:



where F is the left adjoint of the forgetfull functor U and $U \circ F = id$.

We describe the ordinary commutative (semi)- ring associated by the right adjoint functor to the n- fold tensor product $\mathbb{Z} \underset{\mathbb{F}_{\pm 1}}{\otimes} \dots \underset{\mathbb{F}_{\pm 1}}{\otimes} \mathbb{Z}$ (resp. $\mathbb{N} \underset{\mathbb{F}}{\otimes} \dots \underset{\mathbb{F}}{\otimes} \mathbb{N}$). Its elements are (non-uniquely) represented as (F, G, σ), where F, G are finite rooted trees, with maps $F \setminus \partial F, G \setminus \partial G \to \{1, 2, \dots, n\}$, and σ is a bijection of their leaves $\sigma : \partial F \xrightarrow{\sim} \partial G$, and for $\mathbb{Z}$ we have in addition signs $\epsilon : \partial F \to \{\pm 1\}$.