# A Natural Probabilistic Model on the Integers and its Relation to Dickman-Type Distributions and Buchstab's Function

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**Abstract**

Let $\{p_j\}_{j=1}^{\infty}$ denote the set of prime numbers in increasing order, let $\Omega_N \subset \mathbb{N}$ denote the set of positive integers with no prime factor larger than $p_N$ and let $P_N$ denote the probability measure on $\Omega_N$ which gives to each $n \in \Omega_N$ a probability proportional to $\frac{1}{n}$. This measure is in fact the distribution of the random integer $I_N \in \Omega_N$ defined by $I_N = \prod_{j=1}^{N} p_j^{X_{p_j}}$, where $\{X_{p_j}\}_{j=1}^{\infty}$ are independent random variables and $X_{p_j}$ is distributed as $\text{Geom}(1 - \frac{1}{p_j})$. We show that $\frac{\log n}{\log N}$ under $P_N$ converges weakly to the *Dickman distribution*. As a corollary, we recover a classical result from classical multiplicative number theory—*Mertens' formula*, which states that $\sum_{n \in \Omega_N} \frac{1}{n} \sim e^{\gamma} \log N$ as $N \to \infty$.

Let $D_{\text{nat}}(A)$ denote the natural density of $A \subset \mathbb{N}$, if it exists, and let $D_{\text{log-indep}}(A) = \lim_{N\to\infty} P_N(A \cap \Omega_N)$ denote the density of $A$ arising from $\{P_N\}_{N=1}^{\infty}$, if it exists. We show that the two densities coincide on a natural algebra of subsets of $\mathbb{N}$. We also show that they do not agree on the sets of $n^{\frac{1}{s}}$*-smooth numbers* $\{n \in \mathbb{N} : p^+(n) \le n^{\frac{1}{s}}\}$, $s > 1$, where $p^+(n)$ is the largest prime divisor of $n$. This last consideration concerns distributions involving the *Dickman function*. We also consider the sets of $n^{\frac{1}{s}}$*-rough numbers* $\{n \in \mathbb{N} : p^-(n) \ge n^{\frac{1}{s}}\}$, $s > 1$, where $p^-(n)$ is the smallest prime divisor of $n$. We show that the probabilities of these sets, under the uniform distribution on $[N] = \{1, \ldots, N\}$ and under the $P_N$-distribution on $\Omega_N$, have the same asymptotic decay profile as functions of $s$, although their rates are necessarily different. This profile involves the *Buchstab function*. We also prove a new representation for the Buchstab function.