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MINIMUM RANK OF POWERS OF TREES*

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Abstract. The minimum rank of a simple graph G over a field $\mathbb{F}$ is the smallest possible rank among all real symmetric matrices, over $\mathbb{F}$, whose (i, j)-entry (for $i \neq j$) is nonzero whenever ij is an edge in G and is zero otherwise. In this paper, the problem of minimum rank of (strict) powers of trees is studied.

Key words. Graph, Minimum rank, Path, (Strict) Power of a graph, Tree.

AMS subject classifications. 05C50, 15A03.

1. Introduction. A *graph* is a pair $G = (V_G, E_G)$, where V_G is the (finite, nonempty) set of vertices of G and E_G is the set of edges, where an edge is an unordered pair of vertices. A matrix $A \in \mathbb{F}^{n\times n}$ ($\mathbb{F}$ a field) is *symmetric* if $A^T = A$.

For an $n \times n$ symmetric matrix A, the *graph of* A, denoted $\mathcal{G}(A)$, is the graph with vertices $\{1, ..., n\}$ and edges $\{ij : a_{ij} \neq 0, 1 \leq i < j \leq n\}$. Note that for symmetric matrices the diagonal is ignored in determining $\mathcal{G}(A)$. Let

$$\mathcal{S}^{\mathbb{F}}(G) = \{A \in \mathbb{F}^{n\times n} : A^T = A, \mathcal{G}(A) = G\}$$

be the set of symmetric matrices over $\mathbb{F}$ described by a graph G. The *minimum rank of a graph* G over the field $\mathbb{F}$ is defined as $\mathrm{mr}^{\mathbb{F}}(G) = \min\{\mathrm{rank}(A) : A \in \mathcal{S}^{\mathbb{F}}(G)\}$. Given a graph G and a field $\mathbb{F}$, the minimum rank problem is to compute $\mathrm{mr}^{\mathbb{F}}(G)$. The minimum rank problem has received significant attention in the last few years; motivation, recent results, and an extensive bibliography can be found in the survey article [6]. Unless explicitly stated otherwise, $\mathbb{F} = \mathbb{R}$ and we write $\mathcal{S}(G)$ and $\mathrm{mr}(G)$ instead of $\mathcal{S}^{\mathbb{R}}(G)$ and $\mathrm{mr}^{\mathbb{R}}(G)$, respectively.

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In this paper, we study the problem of determining the minimum rank of (strict) powers of paths and trees. This problem was initially investigated by the Minimum Rank Group at the AIM Workshop [1].

In Section 2, we introduce the necessary preliminary results and notation for our discussion. Most of the graph theoretic definitions appear in [5, 10]. In Section 3 we provide results on minimum rank of powers and strict powers of paths, and in Section 4 we give our main results on general trees.

2. Notation and terminology. All the graphs in this paper are *simple graphs*, that is, all graphs are loop-free and undirected. The *order of a graph* G, denoted $|G|$, is the number of vertices of G. If $e = uv \in E_G$, we say that u and v are *endpoints* of e; we also say that u and v are *adjacent*, or that they are *neighbors*. For $w \in V_G$, we denote by $N(w)$ the set of all neighbors of w. Two graphs $G = (V, E)$ and $G' = (V', E')$ are *isomorphic*, and we write $G \cong G'$, whenever there exist bijections $\phi : V \to V'$ and $\psi : E \to E'$, such that $v \in V$ is an endpoint of $e \in E$ if and only if $\phi(v)$ is an endpoint of $\psi(e)$. The *degree* of a vertex v, denoted by $\deg(v)$, is the number of edges with v as endpoint. A vertex v is said to be a *pendant* vertex if $\deg(v) = 1$, and the set of pendant vertices in a graph G will be denoted by $\pi(G)$. A vertex v is said to be a *high-degree vertex* whenever $\deg(v) \geq 3$. A *subgraph* of a graph G is a graph H such that $V_H \subseteq V_G$ and $E_H \subseteq E_G$; the graph $G - e$ denotes the subgraph $(V_G, E_G \setminus \{e\})$ of G. If $W \subseteq V_G$ and $E' = \{uv : u, v \in W, uv \in E_G\}$, the graph (W, E') is referred to as the *subgraph of G induced by W* and is denoted by $G[W]$. The subgraph of G induced by $V_G \setminus \{v\}$ is denoted by $G - v$. A *path* on n vertices is the graph $P_n = (\{v_1, v_2, \ldots, v_n\}, \{e_i : e_i = v_i v_{i+1}, 1 \leq i \leq n-1\})$. A graph G, is *connected* if for every pair $u, v \in V_G$, there is a path joining u with v. A graph $T = (V, E)$ is a *tree* if it is connected and $|V| = n$ and $|E| = n - 1$. A *walk of length* r in a graph (V, E) is an alternating sequence: $v_{i_0}, e_{i_1}, v_{i_1}, e_{i_2}, \ldots, v_{i_{r-1}}, e_{i_r}, v_{i_r}$, of vertices, $v_{i_j} \in V$, and edges $e_{i_j} \in E$, not necessarily distinct, such that $v_{i_{j-1}}$ and v_{i_j} are the endpoints of e_{i_j}, for $j = 1, 2, \ldots, r$. A *complete graph* is a graph whose vertices are pairwise adjacent, a complete graph on n vertices is denoted by K_n. A *clique* in a graph G is a complete subgraph G' of G, that is $G' \cong K_{|G'|}$. A *cut-vertex*, in a connected graph G, is a vertex $v \in V_G$, such that $G - v$ is disconnected. A *block* in a graph is a maximal connected subgraph without a cut-vertex. A *block-clique* graph is a graph in which all its blocks are cliques. A graph G is *bipartite* if $V_G = X \cup Y$, with $X \cap Y = \emptyset$, and such that each edge of G has one endpoint in X and the other in Y. A *complete bipartite graph* is a bipartite graph in which each vertex in X is adjacent to all the vertices in Y; a complete bipartite graph is denoted by K_{n_1,n_2}, where $|X| = n_1$ and $|Y| = n_2$. The complete bipartite graph $K_{n,1}$ is a *star*, usually denoted S_n, where n is the number of vertices. The *union* of graphs $G_1, G_2, \ldots, G_k$, denoted $\bigcup_{i=1}^k G_i$, is the graph $\left(\cup_{i=1}^k V_i, \cup_{i=1}^k E_i\right)$. The *path cover number* of a graph

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G, denoted by $P(G)$, is the minimum number of vertex disjoint induced paths in G that cover all the vertices in V_G. An *(edge) covering* of a graph G is a set of subgraphs $\mathcal{C} = \{G_i, i = 1, ..., k\}$ such that G is the non-disjoint union $G = \bigcup_{i=1}^{k} G_i$. For a given covering $\mathcal{C}$, we let $\nu_{\mathcal{C}}(e)$ denote the number of subgraphs that have e as an edge. A *clique covering* in a graph G is a set of cliques such that each edge of G is contained in at least one of these cliques. The *clique covering number* of G, denoted by $\text{cc}(G)$, is the smallest number of cliques in a clique covering of G; the clique covering number is a well-studied parameter.

The *adjacency matrix* of a graph G is the matrix $\mathcal{A}(G) \in \mathcal{S}(G)$, whose nonzero entries are 1's. The (i, j)-entry of $\mathcal{A}(G)^r$ is the number of walks of length r between vertices i and j, and the (i, j)-entry of $\sum_{i=1}^{r} \mathcal{A}(G)^i$ is the number of walks of length at most r between vertices i and j. The *unit matrix*, E_{ij}, is an $n \times n$ matrix whose (i, j)-entry is 1, and all other entries are 0.

Definition 2.1. Let r be a positive integer and $G = (V_G, E_G)$ a graph. The graph G to the power r is the graph $G^r = (V_G, E_{G^r})$, where $ij \in E_{G^r}$ if and only if there is a walk in G from vertex i to vertex j of length at most r.

Note that Definition 2.1 is the classical definition of power of a graph (see [5, pp. 281]). In our discussion of minimum rank of powers of graphs, we also consider strict powers as in the following.

Definition 2.2. Let r be a positive integer and $G = (V_G, E_G)$ a graph. The graph G to the strict power r is the graph $G^{(r)} = (V_G, E_{G^{(r)}})$, where $ij \in E_{G^{(r)}}$ if and only if there is a walk in G from vertex i to vertex j of length exactly r.

If G is a graph, $\sum_{i=1}^{r} \mathcal{A}(G)^i \in \mathcal{S}(G^r)$, while $\mathcal{A}(G)^r \in \mathcal{S}\left(G^{(r)}\right)$, thus the strict definition parallels the definition of power of the adjacency matrix of a graph. The following results can be found in [6, Corollary 1.5, Observations 1.2, 1.6, 1.7 and 1.8]. Item 3 is a consequence of the work in [2].

Observation 2.3. Let G be a graph.

1. If G is connected, then $\text{mr}(G) = |G| - 1$ if and only if $G = P_{|G|}$;
2. If G is connected and $|G| \geq 2$, then $\text{mr}(G) = 1$ if and only if $G = K_{|G|}$;
3. If $G = K_{n_1,n_2}$, with $n_1, n_2 \geq 1, n_1 + n_2 \geq 3$, then $\text{mr}(G) = 2$;
4. If H is an induced subgraph of G, then $\text{mr}(H) \leq \text{mr}(G)$;
5. If G has connected components $G_1, G_2, \ldots, G_k$, then $\text{mr}(G) = \sum_{i=1}^{k} \text{mr}(G_i)$;
6. If $G = \bigcup_{i=1}^{k} G_i$, then $\text{mr}(G) \leq \sum_{i=1}^{k} \text{mr}(G_i)$;
7. $\text{mr}(G) \leq \text{cc}(G)$.

For a tree T, a graphical parameter (the path cover number) is exploited to

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compute $\mathrm{mr}(T)$.

THEOREM 2.4. [9] *If* T *is a tree, then* $\mathrm{mr}(T) = |T| - P(T)$.

The *rank-spread*, $\mathrm{r}_v(G)$, at a vertex $v \in V_G$ is $\mathrm{r}_v(G) = \mathrm{mr}(G) - \mathrm{mr}(G - v)$ (see [3, 8]). The rank spread of a vertex plays a major role in the computation of the minimum rank of a graph with a cut-vertex. The following result gives a formula for computing the minimum rank of such a graph.

THEOREM 2.5. [3, 8] *Suppose that a graph* G *has a cut-vertex* v *and* $G - v$ *results in* k *components. For* $i \in \{1, 2, \ldots, k\}$, *let* $W_i \subseteq V_G$ *be the vertices of the* i*th component, and* G_i *be the subgraph of* G *induced by* $\{v\} \cup W_i$. *Then*

$$\mathrm{mr}(G) = \sum_{i=1}^{k} \mathrm{mr}\left(G_i - v\right) + \min\left\{\sum_{i=1}^{k} r_v\left(G_i\right), 2\right\}.$$

In some cases, optimal matrices over the field $\mathbb{R}$, which realize the minimum rank of a graph over $\mathbb{R}$, can be used to find optimal matrices over other fields. Since most of the minimum ranks over $\mathbb{R}$ in this paper are realized by nonnegative integer matrices, these optimal matrices over $\mathbb{R}$ are also optimal matrices over some other fields. We note this fact where necessary.

PROPOSITION 2.6. [7] *Over an arbitrary field* $\mathbb{F}$, *the minimum ranks of* K_n, K_{n_1,n_2}, *and* P_n *are realized by* $(0,1)$-*matrices.*

Specifically: $\mathrm{mr}^{\mathbb{F}}\left(K_n\right) = \mathrm{rank}^{\mathbb{F}}\left(\mathcal{A}\left(K_n\right) + I_n\right)$, $\mathrm{mr}^{\mathbb{F}}\left(K_{n_1,n_2}\right) = \mathrm{rank}^{\mathbb{F}}\left(\mathcal{A}\left(K_{n_1,n_2}\right)\right)$, *and* $\mathrm{mr}^{\mathbb{F}}\left(P_n\right) = \begin{cases} \mathrm{rank}^{\mathbb{F}}\left(\mathcal{A}\left(P_n\right)\right), & n \text{ odd}, \\ \mathrm{rank}^{\mathbb{F}}\left(\mathcal{A}\left(P_n\right) + E_{11} + E_{nn}\right), & n \text{ even}. \end{cases}$

The following proposition follows from basic matrix rank inequalities and from item 6 of Observation 2.3.

PROPOSITION 2.7. [4, Proposition 2.9] *Let* $\mathbb{F}$ *be a field and* G *be a graph. Suppose* $\mathcal{C} = \{G_i : i = 1, 2, \ldots, k\}$ *is a covering of* G, *and for each* G_i *there is a diagonal matrix* D_i *with entries in* $\mathbb{F}$ *such that* $\mathrm{rank}^{\mathbb{F}}\left(\mathcal{A}\left(G_i\right) + D_i\right) = \mathrm{mr}^{\mathbb{F}}\left(G_i\right)$. *If* $\mathrm{char}(\mathbb{F})$ *is either* 0 *or a prime* p, *and* $\nu_{\mathcal{C}}(e) \not\equiv 0 \pmod p$ *for each edge* $e \in E_G$, *then*

$$\mathrm{mr}^{\mathbb{F}}(G) \leq \sum_{i=1}^{k} \mathrm{mr}^{\mathbb{F}}\left(G_i\right).$$

In particular, if $\nu_{\mathcal{C}}(e) = 1$ *for every edge* $e \in E_G$ *and* $\mathrm{mr}(G) = \sum_{i=1}^{k} \mathrm{mr}\left(G_i\right)$, *then there is an integer diagonal matrix* D *such that* $\mathrm{mr}(\mathcal{A}(G) + D) = \mathrm{rank}(\mathcal{A}(G) + D)$.

3. Powers of paths. This section contains results relative to the minimum rank of powers of paths, and is divided into two parts. In Subsection 3.1, we focus on usual

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powers of a graph in the sense of Definition 2.1 and in Section 3.2, we concentrate on results based on strict powers of graphs as in Definition 2.2.

3.1. Usual powers of paths. It is clear that G^r is a subgraph of G^{r+1} for all $r \geq 1$, thus it is natural to ask if there is a relationship between $\mathrm{mr}\,(G^r)$ and $\mathrm{mr}\,(G^{r+s})$ whenever $s \geq 1$. See Figure 3.1 for an example of the graph power.

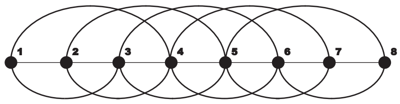

Fig. 3.1. *The graph* P_8^3.

Observation 3.1. For a positive integer m with $1 \leq m \leq n$, and $i \in \{1, 2, \ldots, n-m+1\}$, the induced subgraph of P_n^r on the set of vertices $\{i, i+1, \ldots, i+m-1\}$ is isomorphic to P_m^r.

Note that $\mathrm{mr}\,(P_2) = 1$, because $P_2 \cong K_2$. Thus, for $r \geq 2$, $\mathrm{mr}\,(P_2^r) = 1$.

Theorem 3.2. *For* $n \geq 3$ *and* r *positive integers,*

$$\mathrm{mr}\,(P_n^r) = \begin{cases} n-r & \text{if} \quad 1 \leq r \leq n-2, \\ 1 & \text{if} \quad r \geq n-1. \end{cases}$$

Furthermore, the minimum rank of P_n^r *is realized by a nonnegative integer matrix.*

Proof. From our definition, the vertices of P_n are numbered $1, 2, \ldots, n$, sequentially from a pendant vertex. Note that $ij \in E_{P_n^r}$ if and only if $|i-j| \leq r$. This implies that $\mathrm{mr}\,(P_n^r) \geq n-r$ for r with $1 \leq r \leq n-1$, since the upper right $(n-r) \times (n-r)$ submatrix of any matrix in $\mathcal{S}\,(P_n^r)$ is a full-rank matrix. In addition, $P_n^r \cong K_n$ for $r \geq n-1$, and hence $\mathrm{mr}\,(P_n^r) = 1$ if $r \geq n-1$.

We now prove by induction on n that for $1 \leq r \leq n-2$, $\mathrm{mr}\,(P_n^r) = n-r$. First, if $n = 3$, then $r = 1$ and $\mathrm{mr}\,(P_3) = 2 = n-r$.

Suppose that for $n = k-1$, $\mathrm{mr}\left(P_{k-1}^r\right) = (k-1)-r$, whenever $1 \leq r \leq (k-1)-2$. Also note that if $r = k-2 = (k-1)-1$, then from the case $r \geq n-1$, $\mathrm{mr}\left(P_{k-1}^r\right) = 1$. Let $n = k$, and let r be an integer such that $1 \leq r \leq k-2$. Let H_1 be the subgraph of P_n^r, induced by the set of $n-1$ vertices $\{1, 2, \ldots, n-1\}$, and H_2 the subgraph of P_n^r, induced by the set of $r+1$ vertices $\{n-r, n-r+1, \ldots, n\}$, so that

$$P_n^r \cong (H_1 \cup \{n\}) \bigcup (H_2 \cup \{1, 2, \ldots, n-r-1\})\,.$$

By item 6 in Observation 2.3,

$$\mathrm{mr}\,(P_n^r) \leq \mathrm{mr}\,(H_1 \cup \{n\}) + \mathrm{mr}\,(H_2 \cup \{1, 2, \ldots, n-r-1\}) = \mathrm{mr}\,(H_1) + \mathrm{mr}\,(H_2)\,.$$

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By Observation 3.1, $H_1 \cong P_{n-1}^r$, and $H_2 \cong P_{r+1}^r$ with $1 \leq r \leq k-3$. By the induction hypothesis, $\operatorname{mr}(H_1) = \operatorname{mr}\left(P_{n-1}^r\right) = n-1-r$. Since $r \geq (r+1)-1$, $\operatorname{mr}(H_2) = \operatorname{mr}\left(P_{r+1}^r\right) = 1$. It follows that $\operatorname{mr}(P_n^r) \leq (n-1-r)+1 = n-r$, and consequently, that $\operatorname{mr}(P_n^r) = n-r$.

If G_i is the subgraph of P_n^r induced by the vertices $\{i, i+1, \ldots, i+r\}, i = 1, 2, \ldots, n-r$, then $G_i \cong P_{r+1}^r \cong K_{r+1}$. Also, $\{G_i : i = 1, 2, \ldots, n-r\}$ is an edge covering of G, with $\operatorname{mr}(G_i) = \operatorname{rank}(\mathcal{A}(K_{r+1}) + I_{r+1}) = 1$. As in the proof of Proposition 2.7, let $A_i = [0_{i-1}] \oplus [\mathcal{A}(K_{r+1}) + I_{r+1}] \oplus [0_{n-r-i}]$, where 0_s is the zero matrix of order s. The matrix $A = \sum_{i=1}^{n-r} A_i$ is a nonnegative integer matrix and $\operatorname{rank}(A) = \operatorname{mr}(P_n^r) = n-r$. ☐

Since, for each edge e of P_n^r, $\nu_{\mathcal{C}}(e) \leq r$, the optimal matrix A over $\mathbb{R}$ for Theorem 3.2 is also an optimal matrix over any field $\mathbb{F}$ with $\operatorname{char}(\mathbb{F}) = 0$ or p for some prime $p > r$.

Corollary 3.3. *Let r be a positive integer and p a prime with $p > r$. If $\mathbb{F}$ is a field with* $\operatorname{char}(\mathbb{F}) = 0$ *or p, then the matrix $A = \sum_{i=1}^{n-r} A_i$, as in the proof of Theorem 3.2, satisfies* $\operatorname{rank}^{\mathbb{F}}(A) = \operatorname{mr}^{\mathbb{F}}(P_n^r) = n-r$.

3.2. Strict powers of paths. Although there are similarities between the usual powers and the strict powers of graphs, there are also some interesting differences. For example, the graph $G^{(r)}$ is a subgraph of $G^{(r+2)}$, but not necessarily a subgraph of $G^{(r+1)}$. Note that $G^r = \bigcup_{k=1}^r G^{(r)}$. Recall that from our definition, the vertices of P_n are numbered $1, 2, \ldots, n$, sequentially from a pendant vertex, so the following two observations follow immediately.

Observation 3.4. For a positive integer m with $1 \leq m \leq n$, and $i \in \{1, 2, \ldots, n-m+1\}$, the induced subgraph of $P_n^{(r)}$ on the set of vertices $\{i, i+1, \ldots, i+m-1\}$ is isomorphic to $P_m^{(r)}$.

Observation 3.5. An edge ij is in $E_{P_n^{(r)}}$ if and only if $|i-j| \in \{r, r-2, r-4, \ldots, k\}$, where $k=2$ if r is even and $k=1$ if r is odd.

Proposition 3.6. *Let n and r be a positive integers.*

1. *If r is odd, then $P_n^{(r)}$ is a bipartite graph.*
2. *If r is even, then $P_n^{(r)}$ is a disjoint union of two graphs.*

Proof. If r is odd, then a vertex $i \in V_{P_n^{(r)}}$ is adjacent only to vertices of the opposite parity within distance r. This means that $P_n^{(r)}$ is a bipartite graph.

If r is even, then a vertex $i \in V_{P_n^{(r)}}$ is adjacent only to vertices of the same parity within distance r. This means that $P_n^{(r)}$ is a disjoint union of two graphs. ☐

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Figure 3.2 illustrates the conclusion of Proposition 3.6.

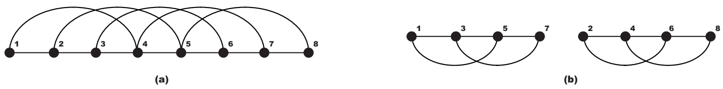

FIG. 3.2. *The graphs* $P_8^{(3)}$ *and* $P_8^{(4)}$

REMARK 3.7. Note that $P_2^{(r)} \cong P_2 \cong K_2$ if r is odd and $P_2^{(r)} \cong K_1 \cup K_1$ if r is even, thus for $r \geq 1$, $\operatorname{mr}\left(P_2^{(r)}\right) = 1$ for r odd and $\operatorname{mr}\left(P_2^{(r)}\right) = 0$ for r even. Also, $P_3^{(r)} \cong P_3$ if r is odd and $P_3^{(r)} \cong K_2 \cup K_1$ if r is even, thus for $r \geq 1$, $\operatorname{mr}\left(P_3^{(r)}\right) = 2$ for r odd, and $\operatorname{mr}\left(P_3^{(r)}\right) = 1$ for r even.

THEOREM 3.8. *For positive integers r, and $n \geq 4$,*

$$\operatorname{mr}\left(P_n^{(r)}\right) = \begin{cases} n - r & \text{if} \quad 1 \leq r \leq n-3, \\ 2 & \text{if} \quad r \geq n-2. \end{cases}$$

Furthermore, $\operatorname{mr}\left(P_n^{(r)}\right)$ *is achieved by a nonnegative integer matrix, and for* $r \geq n-3$*, there is a* $(0,1)$*-matrix which realizes* $\operatorname{mr}\left(P_n^{(r)}\right)$*.*

Proof. From our definition, the vertices of P_n are numbered $1, 2, \ldots, n$, sequentially from a pendant vertex. Notice that $\operatorname{mr}\left(P_n^{(r)}\right) \geq n - r$ for $1 \leq r \leq n-1$, since the upper right $(n-r) \times (n-r)$ submatrix of any matrix in $\mathcal{S}\left(P_n^{(r)}\right)$ is a full-rank matrix.

If $n = 4$, by Theorem 2.4, $\operatorname{mr}(P_4) = 3$. For r odd, $r \geq 3$, we have $P_4^{(r)} \cong K_{2,2}$, and for r even $P_4^{(r)} \cong K_2 \cup K_2$. In either case, $\operatorname{mr}\left(P_4^{(r)}\right) = 2$. By Proposition 2.6, the respective matrices that realize the minimum rank are $\mathcal{A}(P_4) + E_{11} + E_{44}$, $\mathcal{A}(K_{2,2})$, and $\mathcal{A}(K_2) \oplus \mathcal{A}(K_2) + I_4$.

When $r \geq n-2$, and r is odd, the graph $P_n^{(r)}$ is isomorphic to the complete bipartite graph $K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}$. When $r \geq n-2$, and r is even, the graph $P_n^{(r)}$ is isomorphic to the disjoint union, $K_{\lfloor n/2 \rfloor} \cup K_{\lceil n/2 \rceil}$, of two complete graphs. In both cases, $\operatorname{mr}\left(P_n^{(r)}\right) = 2$. By Proposition 2.6, the respective matrices that realize the minimum rank are $\mathcal{A}\left(K_{\lfloor n/2 \rfloor, \lceil n/2 \rceil}\right)$ and $\left(\mathcal{A}\left(K_{\lfloor n/2 \rfloor}\right) \oplus \mathcal{A}\left(K_{\lceil n/2 \rceil}\right)\right) + I_n$.

Suppose $r = n-3$ and $ij \in E_{P_n^{(r)}}$. Then, by Observation 3.5, $|i-j| \in \{n-3, n-5, \ldots, k\}$, with $k = 1, 2$. In either case, this implies that the only edge not in $E_{P_n^{(r)}}$ is $e = 1n$, and thus $P_n^{(r)}$ is isomorphic to $K_{n/2,n/2} - e$, when n is even, and $P_n^{(r)}$ is isomorphic to $\left(K_{\lfloor n/2 \rfloor} \cup K_{\lceil n/2 \rceil}\right) - e$, when n is odd. In both cases, $\operatorname{mr}\left(P_n^{(r)}\right) = 3$;

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the matrices $\mathcal{A}\left(K_{n/2,n/2}\right)-(E_{1n}+E_{n1})+E_{11}+E_{nn}$ and $\mathcal{A}\left(K_{\lfloor n/2\rfloor}\right)\oplus\mathcal{A}\left(K_{\lceil n/2\rceil}\right)+I_n-\left(E_{\lfloor n/2\rfloor+1,\lfloor n/2\rfloor+1}+E_{\lfloor n/2\rfloor+1,n}+E_{n,\lfloor n/2\rfloor+1}+E_{nn}\right)$ realize the minimum ranks, respectively.

Let $1\le r<n-3$ and assume that for $4\le k<n-1$, we have $\operatorname{mr}\left(P_k^{(r)}\right)=k-r$. Assume further, that for $4\le k\le n-1$, there is a nonnegative integer matrix $M\in\mathcal{S}\left(P_k^{(r)}\right)$, with $\operatorname{rank}(M)=\operatorname{mr}\left(P_k^{(r)}\right)$.

For $k=n$, let H_1 be the subgraph of $P_n^{(r)}$ induced by the set of vertices $\{1,2,\dots,n-2\}$, and H_2 the subgraph of $P_n^{(r)}$ induced by the set of vertices $\{n-r-1,n-r,\dots,n\}$, so that

$$P_n^{(r)}\cong(H_1\cup\{n-1,n\})\bigcup(H_2\cup\{1,2,\dots,n-r-2\}).$$

By item 6, in Observation 2.3,

$$\operatorname{mr}\left(P_n^{(r)}\right)\le\operatorname{mr}(H_1\cup\{n-1,n\})+\operatorname{mr}(H_2\cup\{1,2,\dots,n-r-2\})=\operatorname{mr}(H_1)+\operatorname{mr}(H_2).$$

By Observation 3.4, $H_1\cong P_{n-2}^{(r)}$, and $H_2\cong P_{r+2}^{(r)}$, by the induction hypothesis $\operatorname{mr}(H_1)=\operatorname{mr}\left(P_{n-2}^{(r)}\right)=(n-2)-r$, and from the case $r=n-2$, $\operatorname{mr}(H_2)=\operatorname{mr}\left(P_{r+2}^{(r)}\right)=2$. It follows that $\operatorname{mr}\left(P_n^{(r)}\right)\le(n-2-r)+2=n-r$, and consequently, that $\operatorname{mr}\left(P_n^{(r)}\right)=n-r$.

Also by the induction hypothesis there exist nonnegative integer matrices $M_i\in\mathcal{S}(H_i)$, with $\operatorname{rank}(M_i)=\operatorname{mr}(H_i)$, $i=1,2$. Let

$$M=[M_1\oplus 0_2]+[0_{n-r-2}\oplus M_2],$$

clearly M is a nonnegative integer matrix, and $M\in\mathcal{S}\left(P_n^{(r)}\right)$. Furthermore, $n-r\le\operatorname{rank}(M)\le\operatorname{rank}(M_1)+\operatorname{rank}(M_2)=n-r$. ☐

Corollary 3.9. *If $\mathbb{F}$ is a field with* $\operatorname{char}(\mathbb{F})=0$ *or* $\operatorname{char}(\mathbb{F})=p$*, with $p>r$, then the matrix M as in the proof of Theorem 3.8, satisfies* $\operatorname{rank}^{\mathbb{F}}(M)=\operatorname{mr}^{\mathbb{F}}(P_n^{(r)})=n-r$.

4. Strict powers of trees. The next section features results on strict powers of trees, in particular, we relate $\operatorname{mr}\left(T^{(2)}\right)$ to other graph parameters.

Observation 4.1.

1. $S_n^2=K_n$, so $\operatorname{mr}(S_n)=2$, and $\operatorname{mr}(S_n^r)=1, r>1$;
2. $S_n^{(r)}=K_{n-1}\cup K_1$, if r is even, and $S_n^{(r)}=S_n$, if r is odd, so $\operatorname{mr}\left(S_n^{(r)}\right)=1$ if r is even, and $\operatorname{mr}\left(S_n^{(r)}\right)=2$ if r is odd.

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We have already noted that $P_n^{(2)}$ is the disjoint union of two graphs. The following lemma generalizes this notion to trees. Recall that $\pi(T)$ denotes the number of pendant vertices in T.

LEMMA 4.2. *If T is a tree on $n \geq 4$ vertices, with $T \neq S_n$, then $T^{(2)}$ is the disjoint union of two block-clique graphs consisting of a total of $n - \pi(T)$ blocks.*

Proof. Observe that for a pair of vertices z_1 and z_2 in T, there is a path from z_1 to z_2 in $T^{(2)}$ if and only if there is a (unique) path of even length from z_1 and z_2 in T. Let w be a non-pendant vertex in T. For $u, v \in N(w)$, there is the unique path (of length 2) from u to v through w. The graph $Q_w = (N(w), \{uv : u, v \in N(w)\})$, is a maximal clique in $T^{(2)}$. Thus, $T^{(2)}$ consists of the disjoint union of two graphs, one contains Q_w and all the vertices at odd distance from w, and the other contains w and all the vertices at even distance from w.

If none of the vertices in $N(w)$ have neighbors in $T^{(2)}$, outside those in $N(w)$, then the clique Q_w is a component in $T^{(2)}$. Let $v_i \in N(w)$ and $u \notin N(w)$ be adjacent in $T^{(2)}$, and assume that v_i is not a cut-vertex in $T^{(2)}$. Then there is a path in $T^{(2)} - v_i$ from u to $v_j \in N(w)$. But this implies there is a path of even length from u to v_j in T, which is a contradiction, as this path, together with the edges $v_j w$ and $w v_i$, creates a cycle in T. Thus, Q_w forms a block in $T^{(2)}$.

Observe that an edge in $T^{(2)}$ is an edge in at least one Q_w for some nonpendant w. If an edge xy in $T^{(2)}$ is in Q_w and $Q_z, z \neq w$, then $wxzyw$ is a cycle in T, which is a contradiction. Thus, every edge in $T^{(2)}$ is in exactly one Q_w and the intersection of any two Q_w-cliques is a vertex. We have shown that $T^{(2)}$ is the disjoint union of two block-clique graphs.

To count the number of blocks in $T^{(2)}$ we proceed by induction on n. For $n = 4$, the only non-star tree is P_4, and satisfies $P_4^{(2)} = K_2 \cup K_2$, which is a disjoint union of two block-clique graphs consisting of a total of $4 - 2 = 2$ blocks.

Assume that for $|T| = k \leq n - 1$, $T^{(2)}$ is the disjoint union of two block-clique graphs consisting of a total of $n - 1 - \pi(T)$ blocks. Now suppose $|T| = n$, let w be a next-to-pendant vertex in T and U the set of all pendant neighbors of w. The graph $T - U$ is a tree with $|T - U| \leq n - 1$, thus, by induction $(T - U)^{(2)}$ is the disjoint union of two block-clique graphs consisting of $n - |U| - \pi(T - U)$ blocks. We now have two cases:

Case I: w has only one non-pendant neighbor v. In this case, w is a pendant vertex in $T - U$, so the number of blocks in $(T - U)^{(2)}$ is $n - |U| - \pi(T - U) = n - |U| - (\pi(T) - |U| + 1) = n - \pi(T) - 1$. Furthermore, the pendant neighbors of w together with v form an additional clique in $T^{(2)}$, so the total number of blocks in $T^{(2)}$ is $n - \pi(T)$.

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Case II: w has more than one non-pendant neighbor. In this case, w is a non-pendant vertex in $T-U$, so the number of blocks in $(T-U)^{(2)}$ is $n-|U|-\pi(T-U) = n-|U|-(\pi(T)-|U|) = n-\pi(T)$, where the neighbors of w, in $T-U$, form one such block. In $T^{(2)}$, the pendant neighbors of w are adjacent to each other and to the non-pendant neighbors of w. Therefore, no new clique is formed in $T^{(2)}$, only a larger clique, so the total number of blocks in $T^{(2)}$ is $n-\pi(T)$. □

The following lemma provides special cases for paths and stars (note that the second statement is also valid for usual powers) and serve as base cases for induction steps.

LEMMA 4.3. *If T is a path P_n, or a star S_n, where $n \geq 3$, then the following hold.*

1. $\pi(T) - P(T) = 1$*, and*
2. $\operatorname{mr}\left(T^{(2)}\right) = n - \pi(T) = \operatorname{mr}(T) - 1$.

Proof.

1. For $T = P_n$, $\pi(T) = 2$, and $P(T) = 1$. For $T = S_n$, $\pi(T) = n-1$, and $P(T) = n-2$. In both cases $P(T) = \pi(T) - 1$.
2. If $T = P_n$, then by Remark 3.7 and Theorem 3.8, $\operatorname{mr}\left(T^{(2)}\right) = n-2 = n-\pi(T)$. If $T = S_n$, then by Observation 4.1, $\operatorname{mr}\left(T^{(2)}\right) = 1 = n-(n-1) = n-\pi(T)$. □

THEOREM 4.4. *If T is a tree on $n \geq 3$ vertices, then $P(T) \leq \pi(T) - 1$. Furthermore, the equality holds if and only if there are no pairs of adjacent high-degree vertices in T.*

Proof. From Lemma 4.3, the statement is true for paths and stars, thus we may assume $T \neq P_n$, $T \neq S_n$ and proceed by induction on n.

If T has a pendant vertex v that is adjacent to a vertex of degree 2, and $\hat{T} = T - v$, then $\pi(T) = \pi(\hat{T})$, and it is straightforward to see that $P(T) = P(\hat{T})$. Hence, $\pi(T) - P(T) = \pi(\hat{T}) - P(\hat{T}) \geq 1$, where the inequality follows from an induction step.

If every pendant vertex of T is adjacent to a vertex of degree at least 3, and $\hat{T} = T - v$, where v is a pendant vertex, then $\pi(T) = \pi(\hat{T}) + 1$, and $P(T) \leq P(\hat{T}) + 1$; we find readily that $\pi(T) - P(T) \geq \pi(\hat{T}) - P(\hat{T}) \geq 1$, the inequality following from an induction step.

For the second part of the proof, we may assume that $|T| = n \geq 6$, since for $|T| = 2, 3, 4$, and 5 all trees are either paths or stars. We proceed by induction on n.

Suppose that T has two high-degree vertices that are joined by an edge e. Let $\hat{T} = T - e$, and note that $\hat{T}$ is the union of two trees, T_1 and T_2, each on at least

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three vertices. Further, $\pi(T) = \pi(T_1) + \pi(T_2)$, and $P(T) \leq P(T_1) + P(T_2)$. Hence, $\pi(T) - P(T) \geq \pi(T_1) - P(T_1) + \pi(T_2) - P(T_2) \geq 2$.

Now suppose that T has no adjacent pairs of high-degree vertices. Let u be a vertex of high degree, and let $\mathcal{C}$ be a path cover of T of minimum cardinality. Note that some edge e incident with u is not contained in any of the paths in $\mathcal{C}$. If e joins u to a pendant vertex v, let $\tilde{T} = T - v$. Then $\pi(T) = \pi(\tilde{T}) + 1, P(T) = P(\tilde{T}) + 1$, and note that the induction hypothesis applies to $\tilde{T}$. Hence, we have $\pi(T) - P(T) = \pi(\tilde{T}) - P(\tilde{T}) = 1$.

On the other hand, if e joins u to a vertex of degree two, then consider $\overline{T} = T - e$. Note that $\overline{T}$ is the union of two trees, T_1 and T_2, each on at least two vertices, and that the induction hypothesis applies to each of T_1 and T_2. Without loss of generality, $u \in T_1$. Note that $\pi(T) = \pi(T_1) + \pi(T_2) - 1$, since there is exactly one pendant vertex in T_2 that is a non-pendant vertex in T, while all pendant vertices in T_1 are also pendant vertices in T. Also, $P(T) = P(T_1) + P(T_2)$, since e is not contained in any of the paths in the cover $\mathcal{C}$. Hence, we have $\pi(T) - P(T) = \pi(T_1) + \pi(T_2) - 1 - (P(T_1) + P(T_2)) = \pi(T_1) - P(T_1) + \pi(T_2) - P(T_2) - 1 = 1$, the equality following from the induction hypothesis. □

THEOREM 4.5. *If T is a tree on $n \geq 3$ vertices, then* $\operatorname{mr}\left(T^{(2)}\right) \leq n - \pi(T) \leq \operatorname{mr}(T) - 1$. *Furthermore,* $\operatorname{mr}\left(T^{(2)}\right) = \operatorname{mr}(T) - 1$ *if and only if the following two conditions hold:*

1. *T has no pair of adjacent high-degree vertices; and*
2. *each high-degree vertex of T is adjacent to at most two vertices of degree 2.*

Proof. By Lemma 4.3, the statement is true for T a path or a star, otherwise, by Lemma 4.2 there is a clique covering of $T^{(2)}$ of cardinality $n - \pi(T)$ so, from Proposition 2.7, it follows that $\operatorname{mr}\left(T^{(2)}\right) \leq n - \pi(T)$.

From above and Theorem 2.4, $\operatorname{mr}\left(T^{(2)}\right) \leq n - \pi(T) \leq n - P(T) - 1 = \operatorname{mr}(T) - 1$. If there is a pair of high-degree vertices that are adjacent in T, then from Theorem 4.4, we find that $n - \pi(T) < n - P(T) - 1$, so that $\operatorname{mr}\left(T^{(2)}\right) < \operatorname{mr}(T) - 1$.

Suppose now v_0 is a high-degree vertex of T that is adjacent to $k \geq 3$ vertices of degree 2, say u_i, $i = 1, \ldots, k$, and for each $i = 1, \ldots, k$ let v_i be the vertex, distinct from v_0, that is adjacent to u_i. For each non-pendant vertex w of T, let Q_w be the clique in $T^{(2)}$ induced by the vertices (of T) in $N(w)$. Let W denote the collection of all non-pendant vertices of T.

Consider the following union of graphs: $\bigcup_{w \in W, w \neq u_i, 1 \leq i \leq k} Q_w \cup S$, where S is the star in $T^{(2)}$ on the vertices $v_0, v_1, \ldots, v_k$, with v_0 as the center vertex. Observe that this union covers all of the edges of $T^{(2)}$. It now follows from item 6 in Observation 2.3

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that $\mathrm{mr}\left(T^{(2)}\right) \le \sum_{w\in W, w\ne u_i, 1\le i\le k} \mathrm{mr}\left(Q_w\right) + \mathrm{mr}(S) = n-\pi(T)-k+2 < n-\pi(T) \le n-P(T)-1 = \mathrm{mr}(T)-1$.

Suppose now that T is a tree for which both conditions in the statement hold. We claim by induction on n that $\mathrm{mr}\left(T^{(2)}\right) = n-\pi(T) = \mathrm{mr}(T)-1$. By Lemma 4.3 the claim holds when T is a path or a star on $n \ge 3$ vertices. Suppose that the conclusion holds for trees on at most n vertices, that T is on $n+1$ vertices, and that T is neither a path nor a star.

Let u be a high-degree vertex of T that is adjacent to at least one vertex of degree 2, let v_0 be a pendant vertex of T that is adjacent to u, and let $\tilde{T} = T - v_0$. We claim that $\tilde{T}$ satisfies conditions 1 and 2. Certainly condition 1 holds for $\tilde{T}$, and if it were the case that some vertex w of $\tilde{T}$ is adjacent to at least three vertices of degree 2, then necessarily w would have to be adjacent to u (otherwise T would violate condition 2). But then T would violate condition 1, a contradiction. Hence, $\tilde{T}$ satisfies 1 and 2.

Note that $\tilde{T}^{(2)}$ is an induced subgraph of $T^{(2)}$, and so $\mathrm{mr}\left(T^{(2)}\right) \ge \mathrm{mr}\left(\tilde{T}^{(2)}\right)$. By the induction hypothesis, we have $\mathrm{mr}\left(\tilde{T}^{(2)}\right) = n-\pi(\tilde{T}) = n+1-\pi(T)$, so that $\mathrm{mr}\left(T^{(2)}\right) \ge n+1-\pi(T)$. Also, from the first part of the proof, we have $\mathrm{mr}\left(T^{(2)}\right) \le n+1-\pi(T)$, and hence $\mathrm{mr}\left(T^{(2)}\right) = n+1-\pi(T)$. It remains only to show that $\mathrm{mr}(T) = \mathrm{mr}\left(\tilde{T}\right)$, from which we will deduce that $\mathrm{mr}\left(T^{(2)}\right) = \mathrm{mr}\left(\tilde{T}^{(2)}\right) = \mathrm{mr}(\tilde{T})-1 = \mathrm{mr}(T)-1$.

Let $m \ge 3$ denote the degree of the vertex u in T, and for $i \in \{1,\dots,m\}$, let T_i be the branches of T at u, having $|T_1| \ge \cdots \ge |T_m|$ and $T_m = v_0$. For each $i = 1,\dots,m$, let R_i be the subgraph of T induced by $T_i \cup \{u\}$. Evidently, if T_i is a pendant vertex then the rank spread $r_u(R_i) = \mathrm{mr}(R_i) - \mathrm{mr}(T_i) = 1$. Further, if R_i contains a vertex, say w_i of degree 2 adjacent to u, then any path cover of T_i can be extended to a path cover of R_i by including the edge $w_i u$, and hence $r_u(R_i) = 1$ for such an R_i.

By Theorem 2.5, we have $\mathrm{mr}(T) = \sum_{i=1}^m \mathrm{mr}(T_i) + \min\{\sum_{i=1}^m r_u(R_i), 2\}$, and since $\mathrm{mr}(T_i) = 0$ for $i = 3,\dots,m$, it follows that $\mathrm{mr}(T) = \mathrm{mr}(T_1) + \mathrm{mr}(T_2) + \min\{m, 2\} = \mathrm{mr}(T_1) + \mathrm{mr}(T_2) + 2$. Similarly, we find that $\mathrm{mr}(\tilde{T}) = \mathrm{mr}(T_1) + \mathrm{mr}(T_2) + \min\{m-1, 2\} = \mathrm{mr}(T_1) + \mathrm{mr}(T_2) + 2$. Hence, $\mathrm{mr}(T) = \mathrm{mr}(\tilde{T})$, as desired. ∎

We close the paper with a brief discussion of issues arising from the results above.

In view of Theorem 4.5 and the inequality $\mathrm{rank}(A^k) \le \mathrm{rank}(A^{k-1})$, one might suspect that in general $\mathrm{mr}(G^{(r)}) \le \mathrm{mr}(G^{(r-1)})$. However, that is not the case for the star on n vertices, S_n, for instance. It may be interesting to investigate the monotonicity, or lack thereof, of the sequence $\mathrm{mr}(T^{(k)})$ when $T \ne S_n$ is a tree.

We saw in both Corollaries 3.3 and 3.9 that certain nonnegative integer matrices

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attained $\mathrm{mr}^{\mathbb{F}}(P_n^r)$ and $\mathrm{mr}^{\mathbb{F}}(P_n^{(r)})$ when $\mathbb{F}$ is a field of characteristic $p > r$. It may be interesting to determine whether these same matrices realize the minimum rank over fields of characteristic $0 < p \le r$. There may also be some interest in determining whether the minimum ranks (over the reals) of P_n^r or $P_n^{(r)}$ can be realized by $(0,1)$ matrices.

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