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CHANGE OF THE *CONGRUENCE CANONICAL FORM OF 2-BY-2 MATRICES UNDER PERTURBATIONS*

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Abstract. It is constructed the Hasse diagram for the closure ordering on the sets of *congruence classes of 2×2 matrices. In other words, it is constructed the directed graph whose vertices are 2×2 canonical complex matrices for *congruence and there is a directed path from A to B if and only if A can be transformed by an arbitrarily small perturbation to a matrix that is *congruent to B.

Key words. Closure graph, *Congruence canonical form, Perturbations.

AMS subject classifications. 15A21, 15A63, 47A07.

1. Introduction. We study how arbitrarily small perturbations of a 2×2 complex matrix can change its *canonical form for *congruence (matrices A and B are **congruent* if $S^*AS = B$ for a nonsingular S). We construct the closure graph G_2, which is defined for any natural n as follows.

DEFINITION 1.1. The closure graph G_n for *congruence classes of $n \times n$ complex matrices is the directed graph, in which each vertex v represents in a one-to-one manner a *congruence class C_v of $n \times n$ matrices, and there is a directed path from a vertex v to a vertex w if and only if one (and hence each) matrix from C_v can be transformed to a matrix form C_w by an arbitrarily small perturbation.

The graph G_n is the Hasse diagram of the *congruence classes of $n \times n$ matrices with the following partial order: $a \preccurlyeq b$ means that a is contained in the closure of b. Thus, the graph G_n shows how the *congruence classes relate to each other in the affine space of $n \times n$ matrices.

Since each $n \times n$ matrix is uniquely represented in the form $P + iQ$ in which P and Q are Hermitian matrices, G_n is also the closure graph for *congruence classes

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of Hermitian matrix pencils $P + \lambda Q$.

Note that the closure graph G_2 for *congruence, which we construct in Theorem 2.2, is more complicated than the closure graphs for congruence classes of 2-by-2 and 3-by-3 matrices, which were constructed by the authors in [4], since an arrow between *congruence classes in G_2 may depend on the parameters of their matrices.

Unlike perturbations of matrices under congruence and *congruence, perturbations of matrices under similarity and of matrix pencils have been much studied. For a given matrix A, den Boer and Thijsse [3] and, independently, Markus and Parilis [17] described the set of all Jordan canonical matrices such that for each J from this set there exists a matrix that is arbitrarily close to A and is similar to J. Their description was extended to Kronecker's canonical forms of pencils by Pokrzywa [18]. Edelman, Elmroth, and Kågström [7] developed a comprehensive theory of closure relations for similarity classes of matrices, for equivalence classes of matrix pencils, and for their bundles. The software StratiGraph [8] constructs their closure graphs. The closure graph for 2×3 matrix pencils was constructed and studied by Elmroth and Kågström [9].

The term "*congruence orbit" is often used instead of "*congruence class" (see De Terán and Dopico [2]). The problem that we consider can be called "the stratification of orbits of matrices under *congruence" by analogy with the stratification of orbits of matrices under similarity and of matrix pencils [7, 8, 15]. An informal introduction to perturbations of matrices determined up to similarity, congruence, or *congruence is given by Klimenko and Sergeichuk [16].

All matrices that we consider are complex matrices.

2. The closure graph for *congruence classes of 2-by-2 matrices. Define the n-by-n matrices:

$$J_n(\lambda) := \begin{bmatrix} \lambda & 1 & & 0 \\ & \lambda & \ddots & \\ & & \ddots & 1 \\ 0 & & & \lambda \end{bmatrix}, \qquad \Delta_n := \begin{bmatrix} 0 & & & 1 \\ & & \iddots & i \\ & 1 & \iddots & \\ 1 & i & & 0 \end{bmatrix}.$$

We use the following canonical form for *congruence.

PROPOSITION 2.1 ([10, Theorem 4.5.21]). *Each square complex matrix is *congruent to a direct sum, uniquely determined up to permutation of summands, of matrices of the form*

$$\begin{bmatrix} 0 & I_m \\ J_m(\lambda) & 0 \end{bmatrix} \ (0 \neq \lambda \in \mathbb{C},\ |\lambda| < 1), \quad \mu\Delta_n \ (\mu \in \mathbb{C},\ |\mu| = 1), \quad J_k(0). \tag{2.1}$$

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This canonical form obtained in [11] was based on [21, Theorem 3] and was generalized to other fields in [14]. A direct proof that this form is canonical is given in [12, 13].

The vertices of G_n can be identified with the $n \times n$ canonical matrices for *congruence since each *congruence class contains exactly one canonical matrix.

For each $A \in \mathbb{C}^{n\times n}$ and a small matrix $X \in \mathbb{C}^{n\times n}$,

$$(I+X)^*A(I+X) = A + \underbrace{X^*A + AX}_{\text{small}} + \underbrace{X^*AX}_{\text{very small}}$$

and so the *congruence class of A in a small neighborhood of A can be obtained by a very small deformation of the real affine matrix space $\{A + X^*A + AX \mid X \in \mathbb{C}^{n\times n}\}$. (By the local Lipschitz property [20], if A and B are close to each other and $B = S^*AS$ with a nonsingular S, then S can be taken near I_n.) The real vector space

$$T(A) := \{X^*A + AX \mid X \in \mathbb{C}^{n\times n}\}$$

is the tangent space to the *congruence class of A at the point A. The numbers

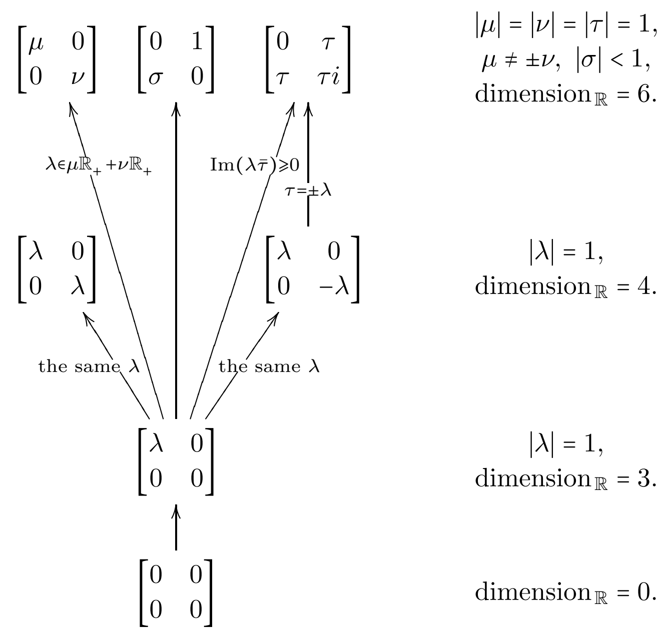
$$\dim_{\mathbb{R}} T(A), \qquad \operatorname{codim}_{\mathbb{R}} T(A) := 2n^2 - \dim_{\mathbb{R}} T(A) \tag{2.2}$$

are called the *dimension* and, respectively, *codimension* over $\mathbb{R}$ of the *congruence class of A.

The following theorem proved in Section 3 is the main result of the paper.

THEOREM 2.2. *The closure graph for *congruence classes of* 2×2 *matrices is*

(2.3)



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in which $\lambda, \mu, \nu, \sigma, \tau \in \mathbb{C}$*,* $\mathbb{R}_+$ *denotes the set of nonnegative real numbers, and* $\operatorname{Im}(c)$ *denotes the imaginary part of* $c \in \mathbb{C}$*. Each *congruence class is given by its canonical matrix, which is a direct sum of blocks of the form* (2.1)*. The graph is infinite: Each vertex except for* $\left[\begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix}\right]$ *represents an infinite set of vertices indexed by the parameters of the corresponding canonical matrix. The *congruence classes of canonical matrices that are located at the same horizontal level in* (2.3) *have the same dimension over* $\mathbb{R}$*, which is indicated to the right.*

The arrow $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \mu & 0 \\ 0 & \nu \end{smallmatrix}\right]$ *exists if and only if* $\lambda = \mu a + \nu b$ *for some nonnegative* $a, b \in \mathbb{R}$*. The arrow* $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} 0 & \tau \\ \tau & i\tau \end{smallmatrix}\right]$ *exists if and only if the imaginary part of* $\lambda\bar{\tau}$ *is nonnegative. The arrow* $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & -\lambda \end{smallmatrix}\right] \to \left[\begin{smallmatrix} 0 & \tau \\ \tau & i\tau \end{smallmatrix}\right]$ *exists if and only if* $\tau = \pm\lambda$*. The arrows* $\left[\begin{smallmatrix} \lambda & 0 \\ 0 & 0 \end{smallmatrix}\right] \to \left[\begin{smallmatrix} \lambda & 0 \\ 0 & \pm\lambda \end{smallmatrix}\right]$ *exist if and only if the value of* λ *is the same in both matrices. The other arrows exist for all values of parameters of their matrices.*

REMARK 2.3. Let M be a 2×2 canonical matrix for *congruence.

- Let N be another 2×2 canonical matrix for *congruence. Each neighborhood of M contains a matrix whose *congruence canonical form is N if and only if there is a directed path from M to N in (2.3) (if $M = N$, then there is the "lazy" path of length 0 from M to N).
- The closure of the *congruence class of M is equal to the union of the *congruence classes of all canonical matrices N such that there is a directed path from N to M (if $M = N$ then the "lazy" path exists).

REMARK 2.4. It is not surprising that $\operatorname{diag}(\lambda, \pm\lambda)$ and $\operatorname{diag}(\mu, \nu)$ ($|\lambda| = |\mu| = |\nu| = 1$ and $\mu \neq \pm\nu$) have different behavior under perturbation: many properties of a nonsingular matrix A with respect to *congruence are determined by its *cosquare $(A^*)^{-1}A$ (see [13, 14, 19]), the *cosquare of $\operatorname{diag}(\lambda, \pm\lambda)$ has a multiple eigenvalue, and the *cosquare of $\operatorname{diag}(\mu, \nu)$ has two distinct eigenvalues.

3. Proof of Theorem 2.2. The following lemma is a weak form of [6, Example 2.1] (which is a special case of [6, Theorem 2.2] about $n \times n$ matrices).

LEMMA 3.1. *Let* A *be any* 2×2 *matrix. Then all matrices* $A + X$ *that are sufficiently close to* A *can be simultaneously reduced by some transformation*

$$\mathcal{S}(X)^*(A + X)\mathcal{S}(X), \qquad \begin{array}{l} \mathcal{S}(X) \textit{ is nonsingular and conti-} \\ \textit{nuous on a neighborhood of zero,} \end{array}$$

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to one of the following forms:

$$\begin{bmatrix}0&0\\0&0\end{bmatrix}+\begin{bmatrix}*&*\\ *&*\end{bmatrix},\qquad \begin{bmatrix}\lambda&0\\0&0\end{bmatrix}+\begin{bmatrix}\varepsilon_\lambda&0\\ *&*\end{bmatrix}\ (|\lambda|=1),$$

$$\begin{bmatrix}\lambda&0\\0&\pm\lambda\end{bmatrix}+\begin{bmatrix}\varepsilon_\lambda&0\\ *&\delta_\lambda\end{bmatrix}\ (|\lambda|=1),\qquad \begin{bmatrix}\lambda&0\\0&\mu\end{bmatrix}+\begin{bmatrix}\varepsilon_\lambda&0\\0&\delta_\mu\end{bmatrix}\ \begin{matrix}(\lambda\neq\pm\mu,\\ |\lambda|=|\mu|=1),\end{matrix}$$

$$\begin{bmatrix}0&1\\ \lambda&0\end{bmatrix}+\begin{bmatrix}0&0\\ *&0\end{bmatrix}\ (|\lambda|<1),\qquad \begin{bmatrix}0&\lambda\\ \lambda&\lambda i\end{bmatrix}+\begin{bmatrix}*&0\\0&0\end{bmatrix}\ (|\lambda|=1).$$

Each of these matrices has the form $A_{\text{can}}+\mathcal{D}$, in which A_{can} is a direct sum of blocks of the form (2.1)*, the $*$'s in $\mathcal{D}$ are complex numbers, all $\varepsilon_\lambda,\delta_\lambda,\delta_\mu$ are either real numbers if $\lambda,\mu\notin\mathbb{R}$ or pure imaginary numbers if $\lambda,\mu\in\mathbb{R}$.* (*Clearly, $\mathcal{D}$ tends to zero as X tends to zero.*) *For each $A_{\text{can}}+\mathcal{D}$, twice the number of its stars plus the number of its entries of the form $\varepsilon_\lambda,\delta_\lambda,\delta_\mu$ is equal to the codimension over $\mathbb{R}$* (*defined in* (2.2)) *of the *congruence class of A_{can}.*

Note that the codimensions of congruence and *congruence classes were calculated in [1, 5] and [2, 6], respectively.

By [22, Part III, Theorem 1.7], the boundary of each *congruence class is a union of *congruence classes of strictly lower dimension, which ensures the following lemma.

LEMMA 3.2. *If $M\to N$ is an arrow in the closure graph G_2, then the *congruence class C_M of M is contained in the closure of the *congruence class C_N of N, and so the dimension of C_M is lower than the dimension of C_N.*

For each vertex M in (2.3), the dimension d_M over $\mathbb{R}$ of the *congruence class of M is indicated in (2.3). It was calculated as follows: By (2.2), $d_M=8-c_M$ in which c_M is the codimension of the *congruence class of M; c_M was taken from Lemma 3.1.

The proof of Theorem 2.2 is divided into two steps.

Step 1: Let us prove that each arrow in (2.3) **is correct.** To make sure that an arrow $M\to N$ is correct, we need to prove that the canonical matrix M can be transformed by an arbitrarily small perturbation to a matrix whose *congruence canonical form is N. Consider each of the arrows of (2.3).

- *The arrows* $\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\to\left[\begin{smallmatrix}\mu&0\\0&\nu\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\to\left[\begin{smallmatrix}\lambda&0\\0&0\end{smallmatrix}\right]$, *and* $\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]\to\left[\begin{smallmatrix}0&\tau\\ \tau&\tau i\end{smallmatrix}\right]$ *are correct.*

Let $A:=\left[\begin{smallmatrix}\mu&0\\0&\nu\end{smallmatrix}\right]$, $\left[\begin{smallmatrix}\lambda&0\\0&0\end{smallmatrix}\right]$, or $\left[\begin{smallmatrix}0&\tau\\ \tau&\tau i\end{smallmatrix}\right]$. Then A is *congruent to εA, in which ε is any positive real number, and each neighborhood of $\left[\begin{smallmatrix}0&0\\0&0\end{smallmatrix}\right]$ contains εA with a sufficiently small ε.

- *The arrow* $\left[\begin{smallmatrix}\lambda&0\\0&0\end{smallmatrix}\right]\to\left[\begin{smallmatrix}\mu&0\\0&\nu\end{smallmatrix}\right]$ (*with given $\lambda,\mu,\nu\in\mathbb{C}$ such that $|\lambda|=|\mu|=|\nu|=1$*)

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exists if and only if $\lambda \in \mu\mathbb{R}_+ + \nu\mathbb{R}_+ = \{\mu a + \nu b | a, b \in \mathbb{R}, a \geqslant 0, b \geqslant 0\}$ *(in particular,* $\left[\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\lambda & 0\\ 0 & -\lambda\end{smallmatrix}\right]$ *exist).*

The arrow $\left[\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mu & 0\\ 0 & \nu\end{smallmatrix}\right]$ exists if and only if there exists an arbitrarily small perturbation

$$\begin{bmatrix}\lambda & 0\\ 0 & 0\end{bmatrix} + E := \begin{bmatrix}\lambda+\varepsilon_{11} & \varepsilon_{12}\\ \varepsilon_{21} & \varepsilon_{22}\end{bmatrix} \quad \text{of} \quad \begin{bmatrix}\lambda & 0\\ 0 & 0\end{bmatrix} \tag{3.1}$$

that is *congruent to $\left[\begin{smallmatrix}\mu & 0\\ 0 & \nu\end{smallmatrix}\right]$. This means that there exists a nonsingular $S = \left[\begin{smallmatrix}x & y\\ z & t\end{smallmatrix}\right]$ such that

$$\begin{bmatrix}\bar{x} & \bar{z}\\ \bar{y} & \bar{t}\end{bmatrix}\begin{bmatrix}\mu & 0\\ 0 & \nu\end{bmatrix}\begin{bmatrix}x & y\\ z & t\end{bmatrix} = \begin{bmatrix}\lambda & 0\\ 0 & 0\end{bmatrix} + E,$$

i.e.,

$$\begin{aligned} \bar{x}x\mu + \bar{z}z\nu &= \lambda + \varepsilon_{11} \qquad & \bar{x}y\mu + \bar{z}t\nu &= \varepsilon_{12}\\ \bar{y}x\mu + \bar{t}z\nu &= \varepsilon_{21} & \bar{y}y\mu + \bar{t}t\nu &= \varepsilon_{22}. \end{aligned} \tag{3.2}$$

For fixed λ, μ, ν and an arbitrarily small ε_{11}, the first equation with unknowns x and z has a solution only if $\lambda \in \mu\mathbb{R}_+ + \nu\mathbb{R}_+$.

Conversely, let $\lambda \in \mu\mathbb{R}_+ + \nu\mathbb{R}_+$. Take $\varepsilon_{11} = 0$ and chose x and z for which the first equality in (3.2) holds. Then take arbitrarily small y, t for which S is nonsingular and get arbitrarily small $\varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22}$ for which the other equalities in (3.2) hold.

- *The arrow* $\left[\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0 & 1\\ \sigma & 0\end{smallmatrix}\right]$ $(|\lambda| = 1,\ |\sigma| < 1)$ *exists for all* λ *and* σ.

The arrow $\left[\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0 & 1\\ \sigma & 0\end{smallmatrix}\right]$ exists if and only if there exists an arbitrarily small perturbation (3.1) that is *congruent to $\left[\begin{smallmatrix}0 & 1\\ \sigma & 0\end{smallmatrix}\right]$. This means that there exists a nonsingular $S = \left[\begin{smallmatrix}x & y\\ z & t\end{smallmatrix}\right]$ such that

$$\begin{bmatrix}\bar{x} & \bar{z}\\ \bar{y} & \bar{t}\end{bmatrix}\begin{bmatrix}0 & 1\\ \sigma & 0\end{bmatrix}\begin{bmatrix}x & y\\ z & t\end{bmatrix} = \begin{bmatrix}\lambda & 0\\ 0 & 0\end{bmatrix} + E,$$

i.e.,

$$\begin{aligned} \bar{x}z + \bar{z}x\sigma &= \lambda + \varepsilon_{11} \qquad & \bar{x}t + \bar{z}y\sigma &= \varepsilon_{12}\\ \bar{y}z + \bar{t}x\sigma &= \varepsilon_{21} & \bar{y}t + \bar{t}y\sigma &= \varepsilon_{22}. \end{aligned} \tag{3.3}$$

Suppose that $\bar{z}x = u + iv$, $\sigma = \alpha + \beta i$, and $\lambda + \varepsilon_{11} = a + bi$, in which $u, v, \alpha, \beta, a, b \in \mathbb{R}$. Then the first equation in (3.3) takes the form $(u - vi) + (u + vi)(\alpha + \beta i) = a + bi$, which gives the system

$$\begin{aligned} (1+\alpha)u - \beta v &= a\\ \beta u + (\alpha - 1)v &= b \end{aligned}$$

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with respect to the unknowns u and v. Its determinant $\alpha^2 + \beta^2 - 1$ is nonzero since $|\sigma| < 1$. Therefore, the first equation in (3.3) holds for some x and z. Taking arbitrarily small y, t for which S is nonsingular, we get arbitrarily small $\varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22}$ for which the other equalities in (3.3) hold.

- *The arrow* $\left[\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0 & \tau\\ \tau & \tau i\end{smallmatrix}\right]$ $(|\lambda| = |\tau| = 1)$ *exists if and only if* $\operatorname{Im}(\lambda\bar\tau) \geqslant 0$.

The arrow $\left[\begin{smallmatrix}\lambda & 0\\ 0 & 0\end{smallmatrix}\right] \to \tau\left[\begin{smallmatrix}0 & 1\\ 1 & i\end{smallmatrix}\right]$ exists if and only if there exists an arbitrarily small perturbation (3.1) that is *congruent to $\tau\left[\begin{smallmatrix}0 & 1\\ 1 & i\end{smallmatrix}\right]$. This means that there exists a nonsingular $S = \left[\begin{smallmatrix}x & y\\ z & t\end{smallmatrix}\right]$ such that

$$\begin{bmatrix}\bar x & \bar z\\ \bar y & \bar t\end{bmatrix}\tau\begin{bmatrix}0 & 1\\ 1 & i\end{bmatrix}\begin{bmatrix}x & y\\ z & t\end{bmatrix} = \begin{bmatrix}\lambda & 0\\ 0 & 0\end{bmatrix} + E,$$

i.e.,

$$\begin{array}{ll}\bar z x + \bar x z + \bar z z i = \bar\tau(\lambda + \varepsilon_{11}) & \bar z y + \bar x t + \bar z t i = \bar\tau\varepsilon_{12}\\ \bar t x + \bar y z + \bar t z = \bar\tau\varepsilon_{21} & \bar t y + \bar y t + \bar t t i = \bar\tau\varepsilon_{22}.\end{array} \tag{3.4}$$

Consider the first equation in (3.4). Since $\bar\tau(\lambda + \varepsilon_{11}) \neq 0$, $z \neq 0$ too. Thus,

$$\operatorname{Im}(\bar\tau(\lambda + \varepsilon_{11})) = \operatorname{Im}(\bar z x + \bar x z + \bar z z i) = \bar z z > 0$$

and so $\operatorname{Im}(\bar\tau\lambda) \geqslant 0$.

Conversely, if $\operatorname{Im}(\bar\tau\lambda) \geqslant 0$, then we put $\varepsilon_{11} = 0$ and take x, z such that the first equation in (3.4) holds. Taking arbitrarily small y, t for which S is nonsingular, we get arbitrarily small $\varepsilon_{12}, \varepsilon_{21}, \varepsilon_{22}$ for which the other equalities in (3.4) hold.

- *The arrow* $\left[\begin{smallmatrix}\lambda & 0\\ 0 & -\lambda\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0 & \tau\\ \tau & \tau i\end{smallmatrix}\right]$ $(|\lambda| = |\tau| = 1)$ *exists if and only if* $\lambda = \pm\tau$.

The arrow $\left[\begin{smallmatrix}\lambda & 0\\ 0 & -\lambda\end{smallmatrix}\right] \to \tau\left[\begin{smallmatrix}0 & 1\\ 1 & i\end{smallmatrix}\right]$ exists if and only if there exists an arbitrarily small perturbation $\left[\begin{smallmatrix}\lambda & 0\\ 0 & -\lambda\end{smallmatrix}\right] + E$ of $\left[\begin{smallmatrix}\lambda & 0\\ 0 & -\lambda\end{smallmatrix}\right]$ that is *congruent to $\tau\left[\begin{smallmatrix}0 & 1\\ 1 & i\end{smallmatrix}\right]$. This means that there exists a nonsingular S such that

$$S^*\tau\begin{bmatrix}0 & 1\\ 1 & i\end{bmatrix}S = \begin{bmatrix}\lambda & 0\\ 0 & -\lambda\end{bmatrix} + E$$

Equating the determinants of both sides, we find that $-\tau^2\det(S^*S)$ is arbitrarily close to $-\lambda^2$. Since

$$\det(S^*S) = \overline{\det S}\det S$$

is a real positive number, $|\tau^2|\det(S^*S)$ is arbitrarily close to $|\lambda^2|$. Since $|\lambda| = |\tau| = 1$, $\det(S^*S)$ is arbitrarily close to 1. Hence, $-\tau^2 = -\lambda^2$, and so $\lambda = \pm\tau$.

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Conversely, let $\lambda = \pm\tau$. Since

$$\begin{bmatrix}1 & 1\\ 1/2 & -1/2\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}1 & 1/2\\ 1 & -1/2\end{bmatrix} = \begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}$$

and

$$\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix}\begin{bmatrix}0 & 1\\ 1 & 0\end{bmatrix}\begin{bmatrix}1 & 0\\ 0 & -1\end{bmatrix} = \begin{bmatrix}0 & -1\\ -1 & 0\end{bmatrix},$$

$\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right]$ is *congruent to $\pm\left[\begin{smallmatrix}0 & 1\\ 1 & 0\end{smallmatrix}\right]$. Its arbitrarily small perturbation $\pm\left[\begin{smallmatrix}0 & 1\\ 1 & \varepsilon i\end{smallmatrix}\right]$ ($\varepsilon \in \mathbb{R}$, $\varepsilon > 0$) is *congruent to $\pm\left[\begin{smallmatrix}0 & 1\\ 1 & i\end{smallmatrix}\right]$ via $\operatorname{diag}(\sqrt{\varepsilon}, 1/\sqrt{\varepsilon})$. Therefore, $\left[\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right] \to \pm\left[\begin{smallmatrix}0 & 1\\ 1 & i\end{smallmatrix}\right]$, and so $\left[\begin{smallmatrix}\lambda & 0\\ 0 & -\lambda\end{smallmatrix}\right] \to \tau\left[\begin{smallmatrix}0 & 1\\ 1 & i\end{smallmatrix}\right]$.

Step 2: Let us prove that we have not missed arrows in (2.3)**.** We write $M \nrightarrow N$ if the closure graph G_2 does not have the arrow $M \to N$; i.e., if each matrix obtained from M by an arbitrarily small perturbation is not *congruent to N. Lemma 3.2 ensures that we need to prove only the absence of the arrows

$$\left[\begin{smallmatrix}\lambda & 0\\ 0 & \pm\lambda\end{smallmatrix}\right] \to \left[\begin{smallmatrix}\mu & 0\\ 0 & \nu\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}\lambda & 0\\ 0 & \pm\lambda\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0 & 1\\ \sigma & 0\end{smallmatrix}\right], \quad \left[\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right] \to \left[\begin{smallmatrix}0 & \tau\\ \tau & \tau i\end{smallmatrix}\right].$$

- $\left[\begin{smallmatrix}\lambda & 0\\ 0 & \pm\lambda\end{smallmatrix}\right] \nrightarrow \left[\begin{smallmatrix}\mu & 0\\ 0 & \nu\end{smallmatrix}\right]$ *and* $\left[\begin{smallmatrix}\lambda & 0\\ 0 & \pm\lambda\end{smallmatrix}\right] \nrightarrow \left[\begin{smallmatrix}0 & 1\\ \sigma & 0\end{smallmatrix}\right]$ ($|\lambda| = |\mu| = |\nu| = 1$, $\mu \neq \pm\nu$, $|\sigma| < 1$).

Suppose that there is an arbitrarily small perturbation $A := \left[\begin{smallmatrix}\lambda & 0\\ 0 & \pm\lambda\end{smallmatrix}\right] + E$ of $\left[\begin{smallmatrix}\lambda & 0\\ 0 & \pm\lambda\end{smallmatrix}\right]$ that is *congruent to $B := \left[\begin{smallmatrix}\mu & 0\\ 0 & \nu\end{smallmatrix}\right]$ or $C := \left[\begin{smallmatrix}0 & 1\\ \sigma & 0\end{smallmatrix}\right]$. Then $A^{-*}A := (A^{-1})^*A$ is similar to $B^{-*}B$ or $C^{-*}C$, which is impossible since the eigenvalues of $A^{-*}A$ are arbitrarily close to $\bar{\lambda}^{-1}\lambda = \lambda^2$, whereas $B^{-*}B = \operatorname{diag}(\mu^2, \nu^2)$ and $C^{-*}C = \operatorname{diag}(\sigma, \bar{\sigma}^{-1})$.

- $\left[\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right] \nrightarrow \left[\begin{smallmatrix}0 & \tau\\ \tau & \tau i\end{smallmatrix}\right]$ ($|\lambda| = |\tau| = 1$).

Let $\left[\begin{smallmatrix}\lambda & 0\\ 0 & \lambda\end{smallmatrix}\right] \to \tau\left[\begin{smallmatrix}0 & 1\\ 1 & i\end{smallmatrix}\right]$; i.e., there exists an arbitrarily small perturbation $A := \left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right] + E$ of $\left[\begin{smallmatrix}1 & 0\\ 0 & 1\end{smallmatrix}\right]$ that is *congruent to $B := \lambda^{-1}\tau\left[\begin{smallmatrix}0 & 1\\ 1 & i\end{smallmatrix}\right]$. This means that there exists a nonsingular S such that

$$S^*\left(\begin{bmatrix}1 & 0\\ 0 & 1\end{bmatrix} + E\right)S = \lambda^{-1}\tau\begin{bmatrix}0 & 1\\ 1 & i\end{bmatrix}$$

Equating the determinants of both sides, we find that

$$r(1+\varepsilon) = -(\lambda^{-1}\tau)^2, \qquad r := \det(S^*S) > 0,$$

in which ε is arbitrarily small. Since $-(\lambda^{-1}\tau)^2$ is fixed and $|\lambda^{-1}\tau| = 1$, we have $(\lambda^{-1}\tau)^2 = -1$, and so $\lambda^{-1}\tau = \pm i$. Then $\operatorname{rank}(B + B^*) = 1$, which is impossible since $A + A^*$ is *congruent to $B + B^*$ and $\operatorname{rank}(A + A^*) = 2$.

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REFERENCES

[1] F. De Terán and F.M. Dopico. The solution of the equation $XA + AX^T = 0$ and its application to the theory of orbits. *Linear Algebra Appl.*, 434:44–67, 2011.

[2] F. De Terán and F.M. Dopico. The equation $XA + AX^* = 0$ and the dimension of *congruence orbits. *Electron. J. Linear Algebra*, 22:448–465, 2011.

[3] H. den Boer and G.Ph.A. Thijsse. Semi-stability of sums of partial multiplicities under additive perturbation. *Integral Equations Operator Theory*, 3:23–42, 1980.

[4] A.R. Dmytryshyn, V. Futorny, B. Kågström, L. Klimenko, and V.V. Sergeichuk. Change of the congruence canonical form of 2-by-2 and 3-by-3 matrices under perturbations. *Linear Algebra Appl.*, to appear. Available at arXiv:1004.3590.

[5] A.R. Dmytryshyn, V. Futorny, and V.V. Sergeichuk. Miniversal deformations of matrices of bilinear forms. *Linear Algebra Appl.*, 436:2670–2700, 2012.

[6] A.R. Dmytryshyn, V. Futorny, and V.V. Sergeichuk. Miniversal deformations of matrices of sesquilinear forms. *Linear Algebra Appl.*, 446:388-420, 2014.

[7] A. Edelman, E. Elmroth, and B. Kågström. A geometric approach to perturbation theory of matrices and matrix pencils. Part II: A stratification-enhanced staircase algorithm. *SIAM J. Matrix Anal. Appl.*, 20:667–699, 1999.

[8] E. Elmroth, P. Johansson, and B. Kågström. Computation and presentation of graph displaying closure hierarchies of Jordan and Kronecker structures. *Numer. Linear Algebra Appl.*, 8:381–399, 2001.

[9] E. Elmroth and B. Kågström. The set of 2-by-3 matrix pencils – Kronecker structures and their transitions under perturbations. *SIAM J. Matrix Anal. Appl.*, 17:1–34, 1996.

[10] R.A. Horn and C.R. Johnson. *Matrix Analysis*, second edition. Cambridge University Press, Cambridge, 2012.

[11] R.A. Horn and V.V. Sergeichuk. Congruence of a square matrix and its transpose. *Linear Algebra Appl.*, 389:347–353, 2004.

[12] R.A. Horn and V.V. Sergeichuk. A regularization algorithm for matrices of bilinear and sesquilinear forms. *Linear Algebra Appl.*, 412:380–395, 2006.

[13] R.A. Horn and V.V. Sergeichuk. Canonical forms for complex matrix congruence and *congruence. *Linear Algebra Appl.*, 416:1010–1032, 2006.

[14] R.A. Horn and V.V. Sergeichuk. Canonical matrices of bilinear and sesquilinear forms. *Linear Algebra Appl.*, 428:193–223, 2008.

[15] S. Johansson. Canonical forms and stratification of orbits and bundles of system pencils. Report UMINF-05.16, Department of Computing Science, Umeå University, Sweden, 2005.

[16] L. Klimenko and V.V. Sergeichuk. An informal introduction to perturbations of matrices determined up to similarity or congruence. *São Paulo J. Math. Sci.*, to appear. Available at arXiv:1311.1144.

[17] A.S. Markus and E.È. Parilis. The change of the Jordan structure of a matrix under small perturbations. *Mat. Issled.*, 54:98–109 1980. English translation: *Linear Algebra Appl.*, 54:139–152, 1983.

[18] A. Pokrzywa. On perturbations and the equivalence orbit of a matrix pencil. *Linear Algebra Appl.*, 82:99–121, 1986.

[19] C. Riehm and M. Shrader-Frechette. The equivalence of sesquilinear forms. *J. Algebra*, 42:495–530, 1976.

[20] L. Rodman. Remarks on Lipschitz properties of matrix groups actions. *Linear Algebra Appl.*, 434:1513–1524, 2011.

[21] V.V. Sergeichuk. Classification problems for system of forms and linear mappings. *Math. USSR-Izv.*, 31:481–501, 1988.

[22] A. Tannenbaum. *Invariance and System Theory: Algebraic and Geometric Aspects*. Lecture Notes in Math., Vol. 845, Springer, 1981.